One-dimensional algebraic cycles on nonsingular cubic fourfolds in $\mathbb{P}^5$

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by

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Abstract

In the thesis we study codimension p algebraic cycles on a $2p$-dimensional nonsingular projective variety $\mathscr{X}$ defined over an uncountable algebraically closed ground field k of characteristic 0. The main result (Theorem 4.7.1 in the thesis) says that, under some weak representability assumptions on the continuous parts of the Chow groups of the variety $\mathscr{X}$ and its nonsingular hyperplane sections $\mathscr{Y}$, the kernel of the Gysin homomorphism from the codimension p Chow group of the very general $\mathscr{Y}$ to the codimension $p+1$ Chow group of $\mathscr{X}$ is countable. As an application, we obtain the following concrete result. Let $\mathscr{X}$ be a nonsingular cubic hypersurface in $\mathbb{P}^5$ over k. Then, for a very general $\mathscr{Y}$, there exists a countable set Ξ of closed points on the Prym variety of the threefold $\mathscr{Y}$, such that, if Σ and Σ' are two linear combinations of lines of the same degree on $\mathscr{Y}$, the one-cycle Σ is rationally equivalent to the one-cycle Σ' on $\mathscr{X}$ if and only if the difference $\Sigma - \Sigma'$, as a point of the Prymian, is an element of Ξ. These results first appeared in the joint preprint [4]. In the thesis we give a detailed exposition of the arguments and methods presented in loc.cit.

Contents

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Chapter 1

Introduction

In algebraic geometry, an algebraic cycle is a linear combination of closed irreducible subvarieties in a given ambient variety X over a field. The coefficients in an algebraic cycle are called multiplicities, which allow to define cycle-theoretic intersections of algebraic cycles inside X. The irreducible components of the set-theoretic intersection of prime cycles are endowed with multiplicities coming up as orders of zeros or poles of the rational functions relevant to this intersection. Given two algebraic cycles A and A', we say that A is rationally equivalent to A' if there exists a positive algebraic cycle Z on the product of $X \times \mathbb{P}^1$ and a big enough positive algebraic cycle B on X, such that the cycle-theoretic intersections $Z(0)$ and $Z(\infty)$ of Z with the fibres over two fundamental points on $\mathbb{P}^1$ coincide with the positive cycles $A + B$ and $A' + B$ on X. In other words, $A + B$ can be cycle-theoretically deformed to $A' + B$ along the cycle Z.

Rational equivalence of algebraic cycles is a fundamental property encoding many important phenomena in algebraic geometry and arithmetic. It is well understood in codimension 1. The classical Abel-Jacobi theorem says that if X is a curve then the algebraic cycles of degree 0 on X are parametrized by an abelian variety, called the Jacobian variety of the curve X. The study of subvarieties of codimension 1 modulo rational equivalence in higher dimension has been completed in the mid of 20th century by proving the existence of Picard schemes extending the Abel-Jacobi theorem for curves (Grothedieck, Altman-Kleiman and others). In contrast, rational equivalence of algebraic cycles in codimension 2 and higher still remains a mystery in algebraic geometry. As we go deep inside X, i.e. as codimension of subvarieties is increasing, the rational equivalence is getting to be more difficult to understand. This happens already in the case of 0-dimensional cycles on algebraic surfaces. The celebrated Bloch's conjecture says that if the geometric genus of a smooth projective complex surface is 0 then the Albanese kernel of X is also trivial, i.e. 0-dimensional cycles modulo rational equivalence on X can be parametrized by the Albanese variety of X, similarly to the Abel-Jacobi theorem for curves. This conjecture was easily proved for surfaces of Kodaira dimension < 2 (Bloch-Kas-Lieberman) but if X is of general type the problem is solved only in few cases (Barlow, Inose-Mizukami, Kimura, Voisin).

Rational equivalence is also important in birational geometry. Recall that birational classification of projective surfaces in characteristic zero has been completed by the classical Italian geometers (Enriques and others) and by Bombieri and Mumford in positive characteristic. The next problem was to understand the Lüroth problem in dimension 3. This problem turned out to be difficult and, after many years since it was stated, it has been successfully resolved by Clemens and Griffiths in [9] and also by Iskovskih and Manin [38]. Roughly speaking, the method of Clemens and Griffiths is as follows. First we look at the continuous part $A^2(X)$ of the Chow group of codimension 2 algebraic cycles modulo rational equivalence on a nonsingular cubic threefold X in $\mathbb{P}^4$ and interpret it as the Griffiths' intermediate Jacobian $J^2(X)$ of X. Then observe that $J^2(X)$ can contain Jacobians of curves only if X is rational. Finally, analyze the theta-divisor on the principally polarized abelian variety $J^2(X)$ to show that there is no Jacobians of curves inside it. Thus, the Clemens-Griffiths method is rooted in rational equivalence of codimension 2 algebraic cycles on the threefold X, which were interpreted in terms of the intermediate jacobian $J^2(X)$. Iskovskih and Manin worked out the counterexample to the Lüroth problem in the case of quartic threefolds. Due to Mumford's theory of Prym varieties, the analytical arguments can be avoided and irrationality of a nonsingular cubic threefold can be proven over an arbitrary ground field of characteristic not 2 or 3, see [26]. The work of Voisin [43] , Colliot-Thelénè and Pirutka [3] is also relevant in this regard.

The above example shows that rational equivalence for algebraic cycles of intermediate codimension can be important in approaching difficult conjectures in birational algebraic geometry, such as the conjecture on non-rationality of a nonsingular cubic fourfold in $\mathbb{P}^5$. The problem is, however, that rational equivalence of algebraic cycles of intermediate codimension on high dimensional varieties is not yet well understood. For example, it is well known that each dimension 1 algebraic cycle on a nonsingular cubic hypersurface in $\mathbb{P}^5$ is rationally equivalent to a linear combination of lines, see [27] and [33]. Yet it seems to be difficult to give a satisfactory answer to the question whether two combinations of lines are rationally equivalent one to another. The purpose of our thesis is to approach a question of this kind via the monodromy argument, essentially used in [40], [41] and more recently in [42][1].

To us Proposition 2.4 in [42] is the departing point. In a nutshell, it means the following. Take for instance a nonsingular projective surface $\mathscr{X}$ over $\mathbb{C}$, embed it into a projective space and consider a smooth hyperplane section $\mathscr{Y}$ of $\mathscr{X}$. For simplicity, assume that $\mathscr{X}$ is regular, that is the local ring of every closed point of $\mathscr{X}$ is regular. Then Voisin's result tells us that, if $\mathscr{Y}$ is very general, the kernel of

[1] see also other papers by C. Voisin

the push-forward homomorphism from the Jacobian J of $\mathscr{Y}$ to the Chow-group of zero-cycles on $\mathscr{X}$ is either the whole Jacobian J or a countable subset in it. And the same phenomena is true for zero-cycles supported on a dimension one linear section of a nonsingular projective variety of arbitrary dimension, loc.cit. In this thesis we address the following questions, in that regard: can one extend Voisin's method to algebraic cycles of codimension $p+1$ supported on a very general hyperplane section of a nonsingular projective variety of dimension $2p$, and can we avoid analytic arguments working over an arbitrary uncountable field of characteristic zero?

The thesis is based on the joint work with V. Guletskii, [4], and represents a detailed exposition of the results and arguments given there. The first two chapters in the thesis aim to give basics on algebraic cycles and introduce the reader into the subtle phenomena relevant to étale monodromy and étale Picard-Lefschetz formula. There is nothing new in the first two chapters, and the reader is advised to read them only in case of necessity. The new results are coming in Chapter 4, and we give a brief synopsis of it here.

In Section 1 of Chapter 4 we generalize the well-known Mumford-Roitman countability result, [28], to algebraic cycles of arbitrary codimension. In Section 4.2 we prove that, for a proper morphism r from one nonsingular projective variety Y to another nonsingular projective variety X, such that codimension p algebraic cycles on Y are represented by an abelian variety, say A, the kernel of the corresponding push-forward homomorphism r_* on Chow groups is a countable union of shifts of an abelian subvariety A_0 in A. In Section 4.4 we study weak representability of continuous parts of codimension p Chow groups in a family. In particular, we explain there the connection between the geometric generic fibre of a family and its very general geometric fibre, in the context of the above abelian variety A_0. In Chapter 4.5 we first consider the case when Y is the geometric generic fibre of a Lefschetz pencil of a nonsingular variety $\mathscr{X}$ embedded into a projective space $\mathbb{P}^m$, and X is the product of $\mathscr{X}$ and the geometric generic point of the base. Being inspired by Voisin's analytic monodromy argument over $\mathbb{C}$, we apply the absolute irreducibility of the étale monodromy action and prove that either A_0 is zero or it is A_1 (if $H^{2p+1}(\mathscr{X})$ vanishes then $A_1 = A$), here A_1 is the abelian variety that arises from the non-vanishing of $H^{2p+1}(\mathscr{X})$, containing A_0. Then, using non-canonical scheme-theoretic isomorphisms between the geometric generic fibre and the very general geometric fibre, we expand the above alternative to all very general fibres for the family of hyperplane sections of $\mathscr{X}$. Using this, and assuming $H^{2p+1}(\mathscr{X}) = 0$, we obtain (Theorem 4.5.2) that, for a very general hyperplane section $\mathscr{Y}$, the kernel of the push-forward homomorphism from $A^p(\mathscr{Y})$ to $A^{p+1}(\mathscr{X})$ is either a countable set, or the whole group $A^p(\mathscr{Y})$. Notice that the assumption $H^{2p+1}(\mathscr{X}) = 0$ is not essential and can be eliminated at low cost. The main result is Theorem 4.7.1 which says that if, moreover, the group $A^{p+1}(\mathscr{Y}_{\bar{\xi}})$ is weakly representable and the group $A^{p+1}(\mathscr{X})$ is not rationally weakly representable, then for a very general $\mathscr{Y}$, the kernel of the homomorphism from $A^p(\mathscr{Y})$ to $A^{p+1}(\mathscr{X})$ is countable.

In the last section we apply the main Theorem 4.7.1 to the case when $\mathscr{X}$ is a nonsingular cubic in $\mathbb{P}^5$. As a concrete application, we obtain the result for cubic fourfold hypersurfaces described in the abstract of the thesis above. Namely, let $\mathscr{X}$ be a nonsingular cubic hypersurface in $\mathbb{P}^5$ over k. The ground field k is uncountable, algebraically closed of characteristic zero (for example, $\mathbb{C}$ or the algebraic closure of $\mathbb{Q}_p$, for a prime p). Recall that $A^3(\mathscr{X})$ is generated by lines, see [33]. For any nonsingular hyperplane section $\mathscr{Y}$ of the fourfold $\mathscr{X}$ let $\mathscr{P}$ be the Prym variety of the threefold $\mathscr{Y}$. Then, for a very general $\mathscr{Y}$, there exists a countable set Ξ of closed points on the principally polarized abelian variety $\mathscr{P}$, such that, if Σ and Σ' are two linear combinations of lines of the same degree on $\mathscr{Y}$, the one-cycle Σ is rationally equivalent to the one-cycle Σ' on $\mathscr{X}$ if and only if the difference $\Sigma - \Sigma'$, as a point of the Prymian $\mathscr{P}$, is an element of Ξ.

Chapter 2

Algebraic cycles

The purpose of this chapter is to recall some needed facts in intersection theory which we use without making references throughout the text. The material below is an elaboration of the first chapter in [13] (using also [18]). Action of correspondences and weak representability is borrowed from [8]. To make the exposition more lucid we provide numerous examples illustrating general principals of the theory. As in [13], a scheme is always an algebraic scheme, i.e. of finite type over a field, unless it is a spectrum of a local ring at a point. A variety is a reduced and irreducible scheme. A subvariety is a closed sub-scheme of a variety which itself is a variety. A point on a variety is a closed point, unless otherwise specified.

2.1 Rational equivalence

Let k be an algebraically closed field of characteristic zero. Consider a variety X that is regular in codimension one, that is the local ring $\mathscr{O}_{X,V}$ of each integral closed subscheme V of codimension one is regular. Now we fix V, let A denote the local ring $\mathscr{O}_{X,V}$ which is an integral domain. Since A is a regular ring it is a DVR. Since the field of rational functions $R(X)$ is the fraction field to A, we get a discrete valuation

$$\mathrm{ord}_V : R(X)^* \to \mathbb{Z} ,$$

for any $a \in A$, $\mathrm{ord}_V(a)$ is nothing but the length of the ring $A/(a)$ considered as a module over A. For any rational function f on X, the value $\mathrm{ord}_V(f)$ is non-zero only for finitely many codimension one integral closed subscheme V of X. Let $U = \mathrm{Spec}(A)$ be an affine open set on which f is regular. Then the complement $X \smallsetminus U$ can be contain only a finite collection of codimension one integral closed subschemes in X and therefore we can ignore it. Now for each V such that $V \cap U$ is non-empty the value $\mathrm{ord}_{V\cap U}(f)$ is non-negative and it is positive precisely when $V \cap U$ is a closed subset of $Z(f)$ of zeros of f on U. But $Z(f)$ can only have a finite number of codimension one integral closed sub-schemes in X.

An algebraic n-cycle on an algebraic scheme X is an element of the free abelian group $\mathscr{Z}_n(X)$ generated by integral closed subschemes of dimension n on

X. We can give the reduced induced subscheme structure on each irreducible closed subset of X. This gives a 1-1 correspondence between integral closed subschemes in X and irreducible closed subsets in X in the Zariski topology. So it is meaningful to say that $\mathscr{Z}_n(X)$ is generated by closed irreducible subsets of dimension n in X. We will write $\mathscr{Z}_*(X)$ for $\oplus \mathscr{Z}_n(X)$.

Take an $n+1$ dimensional integral closed subscheme W of X and consider a rational function f in $R(W)^*$, then we define divisor $\operatorname{div}(f)$ associated to f to be the n-cycle

$$\sum \operatorname{ord}_V(f)V$$

this is well defined as there are only finitely many V such that $\operatorname{ord}_V(f)$ is non-zero.

Two n-dimensional cycles α and α' are said to be rationally equivalent on X if there exists finitely many $n+1$ dimensional integral closed subschemes W_i of X and non-zero rational functions f_i on W_i such that

$$\alpha - \alpha' = \sum \operatorname{div}(f_i) \ .$$

Now for any two non-zero rational functions f, g on X we have that

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$$

this is because ord_V is a homomorphism of groups.

Thus, the cycles rationally equivalent to zero form a subgroup of $\mathscr{Z}_n(X)$. We denote this subgroup by $\mathscr{R}at_n(X)$. The quotient group

$$CH_n(X) = \mathscr{Z}_n(X)/\mathscr{R}at_n(X)$$

is called the Chow group of n-dimensional algebraic cycles modulo rational equivalence. Elements of the Chow group $CH_n(X)$ can be called as cycles classes on X. The following lemma is useful for understanding the nature of Chow groups.

Lemma 2.1.1. *Let X be a smooth variety over an algebraically closed field k and let Y be any variety over k. Let $K = k(Y)$ be the function field on Y and let X_K be the scheme obtained by field extension from k to K. Then $CH^n(X_K)$ is isomorphic to the colimit of the Chow groups $CH^n(X \times_k U)$ over all Zariski open subsets U in Y.*

Proof. See pages 21 - 22 in [7]. □

Later on we will give another equivalent definition of rational equivalence on algebraic cycles, which will shed more geometrical light on this important notion. But to do that we need to study how algebraic cycles behave with regard to morphisms of algebraic varieties over a field.

2.2 Proper push-forward and flat pull-back

Let X be a an algebraic scheme. Let $X_1, \cdots, X_n$ be the irreducible topological components of X. Since for a scheme X, every irreducible closed subset has a unique generic point, we denote by ξ_i the generic points of the irreducible closed sets X_i. Since X_i is not contained in any larger closed irreducible subset of X, we get that the local rings $\mathscr{O}_{X,\xi_i}$ are zero dimensional, that the supremum of the length of chain of prime ideals is zero. Since $\mathscr{O}_{X,\xi_i}$ is zero dimensional and also Noetherian, it is of finite length. The length of $\mathscr{O}_{X,\xi_i}$ is called the geometric multiplicity of X along X_i, and it is denoted by m_i. The fundamental cycle of X is then

$$[X] = \sum m_i X_i$$

as an element of $\mathscr{Z}_*(X)$. If all the irreducible components of X have the same dimension d, then we have that $[X]$ is an element of $\mathscr{Z}_d(X)$. Observe that for an integral scheme V the local ring $\mathscr{O}_{V,\xi}$, the stalk at the generic point ξ of V, is a field. Therefore it is of dimension one, hence the fundamental cycle $[V]$ is just V. Now we define the notion of proper push-forward and flat pullback of algebraic cycles.

Consider $f : X \to Y$ a proper morphism between two schemes. By definition it follows that f is separated and universally closed, so we if we consider a base change of f, then it takes closed subsets into closed subsets. Let V be an integral closed subscheme in X. Then it follows that $W = f(V)$ is an integral closed subscheme in Y, and we have the an embedding f^* induced by f of $k(W)$ into $k(V)$, which is a finite extension if the dimension of V and W are the same [17, 5.5.6]. Now set, $\deg(V/W) = [k(V) : k(W)]$ if $\dim(W) = \dim(V)$ and $\deg(V/W) = 0$ otherwise, where $[k(V) : k(W)]$ denotes the degree of the field extension, then we define

$$f_*[V] = \deg(V/W)[W] \ .$$

This extends linearly to a homomorphism

$$f_* : \mathscr{Z}_n(X) \to \mathscr{Z}_n(Y) \ .$$

If $g : Y \to Z$ is another proper morphism then we have that

$$(g \circ f)_* = g_* \circ f_* \ .$$

Consider the following example showing that the requirement of separatedness is essential here.

Example 2.2.1. Let P be a closed point on $\mathbb{P}^1$, and let $U = \mathbb{P}^1 \setminus \{P\}$. Consider X to be a scheme constructed by pasting of two copies of $\mathbb{P}^1$ along the open subset U, and let $i_1 : \mathbb{P}^1 \to X$ and $i_2 : \mathbb{P}^1 \to X$ be the corresponding embeddings. By definition of glueing of two schemes, a subset Y in X is open in X if and only if $i_1^{-1}(Y)$ is open in $\mathbb{P}^1$ and $i_2^{-1}(Y)$ is open in $\mathbb{P}^1$. Using this and the fact that X is a closure of the image of U in X, it is easy to show that X is integral. Now we identify U with $\mathbb{A}^1$ and consider a rational function r on X defined by the single coordinate in U. Let $P_1 = i_1(P)$, $P_2 = i_2(P)$ and P_0 be the image of the origin of coordinates under the embedding of U into X, then $\mathrm{ord}_{P_1}(r) = \mathrm{ord}_{P_2}(r) = -1$, $\mathrm{ord}_{P_0}(r) = 1$ and $\mathrm{ord}_Q(r) = 0$ for any other point Q on X. So it follows that

$$\mathrm{div}(r) = P_0 - P_1 - P_2$$

and further $P_1 + P_2$ is rationally equivalent to P_0 on X. On the other hand, $f_*(P_1 + P_2) = 2$ is not rationally equivalent to $f_*(P_0) = 1$ on $\mathrm{Spec}(k)$.

Now let us recall the definition of a flat morphism between two schemes. A morphism $f : X \to Y$ is said to be flat if for each point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{Y,y}$ module. Let us consider a flat morphism f from a scheme X to another scheme Y of relative dimension n. Let for a d-dimensional integral closed subscheme V of Y, $f^{-1}(V)$ be the fibred product of X and V over Y. Then we can define the pull-back homomorphism associated to f putting

$$f^*[V] = [f^{-1}(V)] \ ,$$

and we extend f^* by linearity on the whole $\mathscr{Z}_d(Y)$. Since V is an integral closed subscheme and f is a flat morphism of relative dimension n, it follows that $f^{-1}(V)$ is a closed subscheme of X of dimension $d + n$. So we obtain a homomorphism

$$f^* : \mathscr{Z}_d(Y) \to \mathscr{Z}_{d+n}(X) \ .$$

We now head towards proving some fundamental properties of push-forwarding and pulling back. First we will prove a few useful lemma. The first one will be elementary topological, the other one is concerned about flat morphisms of schemes. For any subset S of a topological space X let $\bar{S}$ denote the closure of S in X.

Let W be an irreducible component of X, and let V be the topological closure $\overline{f(W)}$ of $f(W)$ in Y. Then V is an irreducible component of Y. On the other hand, if we suppose that V is an irreducible component of Y, such that $f^{-1}(V) \neq \emptyset$, then there exists an irreducible component W of X with $\overline{f(W)} = V$. Using this, one can show that for any closed subscheme Z of Y

$$f^*([Z]) = [f^{-1}(Z)] \ ,$$

where $[Z]$ and $[f^{-1}(Z)]$ are the fundamental classes of the schemes Z and $f^{-1}(Z)$ respectively, see [13], Lemma 1.7.1.

Let now

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a Cartesian square of schemes over a field, and assume that g is flat and f is proper. As flatness and properness is stable under base change, g' is flat and f' is proper too. Then

$$f'_* g'^* = g^* f_*$$

is the equality of two homomorphisms from $\mathscr{Z}_*(X)$ into $\mathscr{Z}_*(Y')$, see Proposition 1.7 on page 18 in [13]. This property of proper push-forward and flat pull-back is essential, and will be used in the text without special references.

Next, the fact that flat pull-backs preserve rational equivalence is more or less straightforward. Preserving rational equivalence by proper push-forwards is less evident, and we give a detail proof of this property below. The reasoning will be a detailed elaboration of the proof of Proposition 1.4 on pages 12 - 13 in Fulton's book [13], borrowed from [18].

First recall that if E/L is a finite field extension then we have a multiplicative homomorphism

$$N_{E/L} : E^* \to L^*$$

sending any element $\alpha \in E^*$ to the determinant $\det(\psi_\alpha)$, where $\psi_\alpha : E \to E$ is the L-linear operator of the finite-dimensional vector space E over L sending any element $\beta \in E$ to the product $\alpha\beta$. A detailed exposition of norms can be found in [29].

Proposition 2.2.2. *Let $f : X \to Y$ be a proper surjective morphism of integral schemes over a field, so we have the induced embedding of $R(Y)$ into $R(X)$. Let $r \in R(X)^*$ and consider the principal divisor $\mathrm{div}(r)$ of the rational function r as an element in $\mathscr{Z}^1(X)$. Then*

$$f_*(\mathrm{div}(r)) = \mathrm{div}(N_{R(X)/R(Y)}(r))$$

if X and Y are of the same dimension, and

$$f_*(\mathrm{div}(r)) = 0$$

otherwise.

Proof. Consider first the case when $\dim(X) = \dim(Y)$. Suppose that f is finite. Therefore it is quasi-finite. Let V be an integral closed subscheme of codimension one in Y, let $W_1, \cdots, W_n$ are integral closed subschemes of X of codimension 1 such that $f(W_i) = V$.

To prove the required formula

$$f_*(\mathrm{div}(r)) = \mathrm{div}(N(r))$$

we have to prove that

$$\sum_i \mathrm{ord}_{W_i}(r)[R(W_i) : R(V)] = \mathrm{ord}_V(N(r)) .$$

This is because of the following. Suppose we take an r, a non-zero rational function on X, then consider the div(r). It can be expressed in the form $\sum_i n_i W_i$, applying f_* we get that

$$f_*[\mathrm{div}(r)] = f_*\left[\sum \mathrm{ord}_{W_i}(r) W_i\right] = \sum \mathrm{ord}_{W_i}(r)[R(W_i) : R(f(W_i))] f(W_i)$$

this sum can be decomposed in the following way, we collect all W_i's such that $f(W_i) = V$. Then we get

$$\sum_{W_i} \sum_{f(W_i)=V} \mathrm{ord}_{W_i}(r)[R(W_i) : R(V)]V$$

to prove that this is equal to

$$\mathrm{div}(N(r))$$

we have to prove that

$$\mathrm{ord}_V(N(r)) = \sum \mathrm{ord}_{W_i}(r)[R(W_i) : R(V)] \ .$$

Let η be the generic point of the closed integral subscheme V in Y. Since f is finite it is affine. So for any affine neighborhood $N = \mathrm{Spec}(A)$ of η the pre-image $f^{-1}(N)$ is affine, denote it by $\mathrm{Spec}(B)$. Since f is finite, we have that B is a finitely generated A-module. Let $\mathfrak{p}$ be the prime ideal in A corresponding to the subscheme $V \cap N$, also let $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ be the prime ideals in B corresponding to the subschemes $W_1 \cap M, \cdots, W_n \cap M$. Now the ring

$$B_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes B$$

is a domain and also it is finitely generated over $A_{\mathfrak{p}}$ since B is finitely generated over A. The field of fraction of $A_{\mathfrak{p}}$ is $R(Y)$, since the field of fraction of B is $R(X)$ we get that the field of fraction for $B_{\mathfrak{p}}$ is $R(X)$. Corresponding to $q_1, \cdots, q_n$ in B we have the prime ideals

$$q_1^e, \cdots, q_n^e$$

in $B_{\mathfrak{p}}$. The localisation $B_{\mathfrak{p}}$ at $\mathfrak{q}_i^e$ is nothing but $B_{\mathfrak{q}_i}$. Let r be in $B_{\mathfrak{p}}$. Let r also denote the multiplication by r in $B_{\mathfrak{p}}$. By lemma $A.2.2$ in [13] we get that

$$e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{p}}) = \sum e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{p}_{\mathfrak{q}_i^e}}) = \sum e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{q}_i}) \ .$$

By the lemma $A.2.3$ in [13]

$$e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{q}_i}) = d_i . e_{B_{\mathfrak{q}_i}}(r, B_{\mathfrak{q}_i})$$

where d_i is the degree $[R(W_i) : R(V)]$. Since $V, W_1, \cdots, W_n$ are integral we have $A_{\mathfrak{p}}$, $B_{\mathfrak{q}_i}$'s are all domains. Therefore we have

$$e_{B_{\mathfrak{q}_i}}(r, B_{\mathfrak{q}_i}) = l(B_{\mathfrak{q}_i}/rB_{\mathfrak{q}_i}) = \mathrm{ord}_{W_i}(r) \ .$$

By lemma $A.3$ in [13] we have

$$e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{p}}) = \mathrm{ord}_{A_{\mathfrak{p}}}(\det(\psi_r)) = \mathrm{ord}_V(N(r))$$

here ψ_r is the multiplication by r, which is an automorphism of $R(X)$ over $R(Y)$. Collecting all the above equalities we get that

$$\sum \mathrm{ord}_{W_i}(r)[R(W_i) : R(V)] = \sum e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{q}_i}) = e_{A_{\mathfrak{p}}}(r, B_{\mathfrak{p}}) = \mathrm{ord}_V(N(r)) \ .$$

In the general case when f is not finite, we use the fact that if we have a proper surjective morphism $f : X \to Y$ of varieties then f factors through a morphism $g : Y' \to Y$, g is finite and $f' : X \to Y'$ has connected fibers. If dimension X is equal to dimension of Y then there is an open set inside X which maps isomorphically onto an open subset of Y' [[16], 4.3.1, 4.4.2]. Let U be the open set which is mapped isomorphically to $f'(U)$. Since any subvariety V of U extends to $\bar{V}$ of X, it is enough to prove that

$$f_*([\mathrm{div}(\bar{r})]) = \mathrm{div}(N(\bar{r}))$$

where $\bar{r}$ is the rational function on U, corresponding to the rational function r in $k(X)$, this always exists since for any non-empty open set U of X, $k(U)$ is isomorphic to $k(X)$. To prove the result in U it is enough to consider it on $f'(U)$. Since f' is an isomorphism on U. But g is finite on $f'(U)$ so we are reduced to the case of finite maps as above.

Now we consider the case $\dim(Y) < \dim(X)$. Start with $Y = \mathrm{Spec}(k)$ and $X = \mathbb{P}^1_k$. Then $k(X)$ is $k(t)$ where t is $\frac{x_1}{x_0}$. Since the order functions are homomorphisms, we can take r to be an irreducible polynomial of degree d inside $k[t]$. Let P be the prime ideal generated by the polynomial r. Since r belongs to P, we have,

$$\mathrm{ord}_P(r) = 1 \ .$$

Also since r is a polynomial in $\frac{x_1}{x_0}$ it can be written as,

$$r = \frac{a_0x_1^d + a_1x_1^{d-1}x_0 + \cdots + a_dx_0^d}{x_0^d} \ .$$

Therefore r has non-zero order along prime ideal defined by the point $P_\infty = (0 : 1)$. The prime ideal at this point is generated by $1/t$. So the uniformizing parameter at the point infinity is $s = 1/t$. Then multiplying r with s^d we have,

$$rs^d = \left(\frac{a_0x_1^d + a_1x_1^{d-1}x_0 + \cdots + a_dx_0^d}{x_0^d}\right)\left(\frac{x_0^d}{x_1^d}\right) \ ,$$

which is

$$= a_0 + \frac{a_1x_0}{x_1} + \cdots + \frac{a_dx_0^d}{x_1^d} \ ,$$

and it does not belong to the ideal generated by $s = 1/t = x_0/x_1$, therefore rs^d is a unit at the point P_∞. Hence we have,

$$\mathrm{ord}_{P_\infty}(rs^d) = 0 \ ,$$

since order is a homomorphism we get that,

$$\mathrm{ord}_{P_\infty}(r) = -\mathrm{ord}_{P_\infty}(s^d) = -d \ .$$

Therefore we obtain,

$$[\mathrm{div}(r)] = [P] - d[P_\infty] \ .$$

Now we have $k(P)$ isomorphic to $k[t]/P$, since P is generated by r, which is a polynomial of degree d,

$$k[t]/P \ ,$$

is a degree d finite extension of the field k. Also we have,

$$k(P_\infty) = k \ ,$$

as P_∞ is a k-rational point. Therefore we get,

$$f_*[\mathrm{div}(r)] = f_*[P] - f_*[dP_\infty] \ ,$$

which is same as,

$$[k(P) : k][Y] - d[Y] = d[Y] - d[Y] = 0 \ .$$

If we have $\dim Y$ is less than $\dim X - 1$ then for any co-dimension one integral subscheme V in X we have $\dim(f(V))$ is less than $\dim(V)$, so by the definition of proper push-forward we get $f_*([V])$ is zero. This is why it is enough to consider the case when

$$\dim Y = \dim X - 1 \ .$$

Let η be the generic point of Y and we consider the following Cartesian square.

$$\begin{array}{ccc} X_\eta & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \eta & \xrightarrow{g} & Y \end{array}$$

The morphism g' is flat as it is a base change of the flat morphism

$$g : \eta \to Y \ .$$

The pull-back g'^* is defined as follows. Let W be a closed integral subscheme of X of codimension one. Let ξ be the generic point of W. Then since f is proper and surjective we can prove that $f(\xi) = \eta$. Take any open set V of Y. Consider $f^{-1}(V)$, it is an open set in X, therefore $f^{-1}(V) \cap W$ will contain ξ, as it is the generic point of W. Then $f(\xi)$ will be contained in V, therefore we get that $f(\xi)$ is the generic point of Y. Since generic point is unique we get that $f(\xi) = \eta$. Now we define

$$g'^{*}[W] = \xi \ .$$

One can verify that

$$\begin{array}{ccc} \mathscr{Z}^1(X_\eta) & \xleftarrow{\ g'^{*}\ } & \mathscr{Z}^1(X) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathscr{Z}^0(\eta) & \xleftarrow{\ g^{*}\ } & \mathscr{Z}^0(Y) \end{array}$$

is commutative. Observe that g^* is an isomorphism of groups, each of them are isomorphic to $\mathbb{Z}$. Therefore we get that

$$f_*(\mathrm{div}(r)) = 0$$

if and only if

$$g^*(f_*(\mathrm{div}(r))) = 0 \ ,$$

which is same as showing

$$f'_* g'^{*}(\mathrm{div}(r)) = 0 \ .$$

Since the map f is surjective there exists a point χ in X such that $f(\chi)$ is equal to η. Now suppose that χ is not a generic point of X, then there exists an open set U such that U does not contain χ. Therefore we get that $f(U)$ does not contain η. Since f is proper and surjective and U open it follows that $f(U)$ is open. Hence we get that η does not belong to the open set $f(U)$, which is a contradiction to the fact that η is a generic point. Therefore χ is the generic point on X which is mapped to η. Consequently we get that χ is in X_η and hence X_η is dense in X, therefore the functions fields $R(X)$ and $R(X_\eta)$ are the same and hence X, X_η are birationally equivalent. Moreover we have

$$g'^{*}(\mathrm{div}(r)) = \mathrm{div}(r)$$

where the first $\mathrm{div}(r)$ is a principal divisor on X and the second one is a principal divisor on X_η. Then we get that

$$f'_* g'^{*}(\mathrm{div}(r)) = f'_*(\mathrm{div}(r)) \ .$$

Therefore $f_*(\mathrm{div}(r)) = 0$ if and only if $f'_*(\mathrm{div}(r)) = 0$ here $\mathrm{div}(r)$ is a principal divisor on X_η. Therefore we are reduced to the case $X_\eta \to \eta$, and without loss of generality we can assume that $Y = \mathrm{Spec}(k)$ and X is a curve over Y. Now choose a normalisation $h : \widetilde{X} \to X$ of the curve X. Let us take the map $h : \widetilde{X} \to X$ and choose a finite map $g : \widetilde{X} \to \mathbb{P}^1_K$, let p be the structural morphism from $\mathbb{P}^1_K$ to $\operatorname{Spec}(K)$. Since $\operatorname{Spec}(K)$ is just one point we obtain that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^1 & \xrightarrow{p} & \operatorname{Spec}(k) \end{array}$$

whence $p \circ g = f \circ h$. Let $\widetilde{r}$ be the image of r under the isomorphism $k(X) \cong k(\widetilde{X})$. Now $\operatorname{div}(r)$ is equal to $h_*(\operatorname{div}(\widetilde{r}))$ this follows from the fact that

$$\operatorname{ord}_V(r) = \sum \operatorname{ord}_{\widetilde{V}}(\widetilde{r})[k(\widetilde{V}) : k(V)]$$

now

$$\operatorname{div}(r) = \sum \operatorname{ord}_V(r) V$$

replace

$$\operatorname{ord}_V(r) = \sum \operatorname{ord}_{\widetilde{V}}(\widetilde{r})[k(\widetilde{V}) : k(V)]$$

then we get

$$\sum_V \sum_{h(\widetilde{V}=V)} \sum \operatorname{ord}_{\widetilde{V}}(\widetilde{r})[k(\widetilde{V}) : k(V)]$$

the above is

$$h_*([\operatorname{div}(\widetilde{r})])$$

and is equal to $\operatorname{div}(r)$. Since

$$f_* \circ h_*(\operatorname{div}(\widetilde{r})) = p_* \circ g_*(\operatorname{div}(\widetilde{r}))$$

since p is from $\mathbb{P}^1_K$ to $\operatorname{Spec} K$, we get that

$$p_*[g_*(\operatorname{div}(\widetilde{r}))] = 0 \ .$$

The proposition is proved. □

2.3 Second definition of rational equivalence

Let X be a scheme over a field k, and take V be an $n+1$-dimensional integral closed subscheme in $X \times \mathbb{P}^1$, that is dominant over $\mathbb{P}^1$, so we have that the composition f of the closed embedding

$$V \to X \times \mathbb{P}^1$$

with the projection

$$q : X \times \mathbb{P}^1 \to \mathbb{P}^1$$

is dominant. Let

$$P : \mathrm{Spec}(k) \to \mathbb{P}^1$$

be a k-rational point on $\mathbb{P}^1$, that lies in the image of f, and let $f^{-1}(P)$ be the scheme-theoretic fibre of the morphism

$$f : V \to \mathbb{P}^1$$

over P. Let $X \times \{P\}$ be the scheme-theoretic pre-image of the point P with respect to the projection q. Now f is defined as the composition of the closed embedding

$$V \to X \times \mathbb{P}^1$$

with the projection q, so we have the induced closed embedding of the fibre $f^{-1}(P)$ into the fibre $X \times \{P\}$. Observe that the latest fibre is naturally isomorphic to the scheme X, so we get the closed embedding

$$f^{-1}(P) \to X \ .$$

To emphasize the fact that $f^{-1}(P)$ is closely embedded into V and into X respectively at the same time, we denote by $V(P)$ the scheme $f^{-1}(P)$ being embedded into X, and let $f^{-1}(P)$ denote the scheme-theoretical pre-image embedded into V. Clearly, $V(P)$ is an n-dimensional cycle on X.

As above, we make a choice of the system of coordinates in $\mathbb{P}^1$ in a way, such that the points $(0 : 1)$ and $(1 : 0)$ are both in the image of f. Then f gives us a non-zero rational function r_f on V, such that

$$[\mathrm{div}(r_f)] = [f^{-1}(0)] - [f^{-1}(\infty)] \ .$$

Let

$$g : V \to X$$

denote the composition of the closed embedding

$$V \to X \times \mathbb{P}^1$$

with the projection

$$p : X \times \mathbb{P}^1 \to X \ .$$

It is easy to verify that the latest equality implies the equality

$$[V(0)] - [V(\infty)] = g_*[\mathrm{div}(r_f)]$$

in $\mathscr{Z}_n(X)$.

Now we can prove the following proposition bringing new understanding of rational equivalence of algebraic cycles. The proof below is borrowed from [13] and [18].

Proposition 2.3.1. *Given an algebraic cycle $A \in \mathscr{Z}_n(X)$, it is rationally equivalent to zero if and only if there exist a finite collection of $n+1$-dimensional varieties $V_1, \ldots, V_m$ in $X \times \mathbb{P}^1$, each of which is dominant over $\mathbb{P}^1$, such that, after appropriate choice of coordinates in $\mathbb{P}^1$, one has*

$$A = \sum_{i=1}^{m} ([V_i(0)] - [V_i(\infty)])$$

in $\mathscr{Z}_n(X)$.

Proof. Let A be a cycle in $\mathscr{Z}_n(X)$ such that

$$A = \sum_i [V_i(0)] - [V_i(\infty)]$$

for some $n+1$ dimensional subvarieties $V_1, \cdots, V_m$ of $X \times \mathbb{P}^1$, each of the above mentioned subvarieties is dominant over $\mathbb{P}^1$. We know that

$$[V_i(0)] - [V_i(\infty)] = g_{i*}[\mathrm{div}(r_{f_i})]$$

where g_i, f_i's are constructed as above. By proposition 2.2.2 it follows that, each of the algebraic cycles $g_{i*}[\mathrm{div}(r_{f_i})]$ is either zero or the principle divisor of the norm. So we have, in both cases it is rationally equivalent to zero.

Conversely let us consider A in $\mathscr{Z}_n(X)$, a cycle rationally equivalent to zero. We have to show that A can be expressed as a sum

$$\sum_i [V_i(0)] - [V_i(\infty)] \,.$$

Without loss of generality take A to be of the form

$$[\mathrm{div}(r)]$$

for some rational function r in $R(W)$, for some $n+1$ dimensional subvariety W in X. Let

$$f : W \dashrightarrow \mathbb{P}^1$$

be the rational morphism induced by the rational function r. Let U be the open set in W where f is regular. Then the morphism

$$f : U \to \mathbb{P}^1 \,.$$

is regular. Let Γ be the scheme theoretical image of the composition

$$c : U \to U \times \mathbb{P}^1 \to W \times \mathbb{P}^1 \to X \times \mathbb{P}^1 \,.$$

The first morphism in the above is constructed as follows. We have the following morphisms

$$\mathrm{id} : U \to U$$

and

$$f : U \to \mathbb{P}^1$$

then by the universal property of the fiber product we have a morphism from U to $U \times \mathbb{P}^1$. Let $V = \Gamma_{\text{red}}$ be the reduction of Γ, which is a closed integral subscheme of $X \times \mathbb{P}^1$. We observe that the underlying topological space $|V|$ of the scheme V is the closure of the graph of the morphism

$$f : U \to \mathbb{P}^1 \ .$$

Observe that the composition of c with the projection

$$X \times \mathbb{P}^1 \to X$$

is the same as composition of

$$U \to W$$

and

$$W \to X \ ,$$

therefore the universal properties of the scheme theoretical image and the morphism

$$V \to \Gamma$$

gives the morphism

$$h : V \to W$$

such that the following square

$$\begin{array}{ccc} V & \longrightarrow & X \times \mathbb{P}^1 \\ {\scriptstyle h}\downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

commutes. This is because of the following. We have a morphism from V to Γ. Also we have a morphism from U to Γ, so by the universal property of V there exist a morphism $V \to U$ and composing this with $U \to W$ we get the morphism

$$V \to W \ .$$

Since the diagram

$$\begin{array}{ccc} U & \longrightarrow & X \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

is commutative we get that

$$\begin{array}{ccc} V & \longrightarrow & X \times \mathbb{P}^1 \\ {\scriptstyle h}\downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

is commutative.

It follows that h is proper, as it is a composition of proper morphisms and it is surjective . Now the morphism

$$f : U \to \mathbb{P}^1$$

is dominant, therefore the composition of the closed embedding

$$V \to X \times \mathbb{P}^1$$

with the projection

$$X \times \mathbb{P}^1 \to \mathbb{P}^1$$

is dominant. We consider U as an open subset of V, then the morphism

$$f : U \to \mathbb{P}^1$$

gives a rational function r_f in $R(V)^*$. We know that

$$g_*[\mathrm{div}(r_f)] = [V(0)] - [V(\infty)]$$

where g is the composition of the closed embedding of V into $X \times \mathbb{P}^1$ and the projection $X \times \mathbb{P}^1 \to X$. Since h maps V to W bi-rationally we have that

$$h_*(\mathrm{div}(r_f)) = \mathrm{div}(N_{R(V)/R(W)}(r)) = \mathrm{div}(r)$$

by proposition 2.2.2. Then we get

$$A = \mathrm{div}(r) = h_*(\mathrm{div}(r_f)) = g_*(\mathrm{div}(r_f)) = [V(0)] - [V(\infty)]$$

the middle equality follows from the commutativity of the diagram

$$\begin{array}{ccc} V & \longrightarrow & X \times \mathbb{P}^1 \\ {\scriptstyle h}\downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

This completes the proof. □

2.4 Weak representability of algebraic cycles

This section is devoted to recall and to discuss the notion of intersection product of two algebraic cycles, correspondences, the weak representability for Chow groups and the Weil cohomology theory of algebraic varieties defined over an algebraically closed field k. For simplicity, we keep assuming that the characteristic of the ground field is zero. This is enough for proving the results of the thesis.

Lets start with the definition of intersection product. Let k be a field and let X be a non-singular, irreducible, projective variety defined over $\mathrm{Spec}(k)$. Let V and W be irreducible subvarieties in X of codimension i and j respectively. Then we write $V \cap W$ as a finite union $\cup_l Z_l$ of irreducible subvarieties $Z_l \subset X$. Suppose that the intersection of V, W at Z_l is proper, that is the codimension of Z_l is $i+j$ for all l. In that case we define the intersection multiplicity of V and W at Z_l as in [19], page 427. We denote this intersection multiplicity by

$$i(V \cdot W; Z_l)\ .$$

Now define the intersection product of V and W as

$$V \cdot W = \sum_l i(V \cdot W; Z_l) Z_l\ ,$$

where in the above formula Z_l stands for the algebraic cycle defined by the irreducible subvariety Z_l of X.

Now lets define the notion of correspondences between smooth, projective, irreducible algebraic varieties over spectrum of a field k. Let X and Y be two smooth projective irreducible varieties over $\mathrm{Spec}(k)$. Let X be of dimension d. A correspondence

$$T \in Corr(X, Y)$$

from X to Y is an element of the Chow group $CH^n(X \times Y)$ for some $n \geq 0$. That is we have $Corr(X, Y)$ equals $CH^*(X \times Y)$, where $CH^*(X \times Y)$ denotes the Chow ring of $X \times Y$, where the ring structure is induced by the intersection product defined above. Let T^t denote the transpose of T, which is an element of $Corr(Y, X)$.

Given a correspondence T in $CH^n(X \times Y)$ we define the homomorphism

$$T_* : CH^i(X) \to CH^{i+n-d}(Y)$$

given by the formula

$$T(Z) = (\mathrm{pr}_Y)_*(T \cdot (Z \times Y))$$

but we can only define the above homomorphism, for those T in $CH^*(X \times Y)$ such that the intersection product $T \cdot (Z \times Y)$ is defined on $X \times Y$, that is the intersection of the irreducible components of the supports of T and $Z \times Y$ is proper. Given two correspondences f from X to Y and g from Y to Z, we can also define their composition given by

$$g \circ f = (\mathrm{pr}_{13})_*(\mathrm{pr}_{12}^* f \cdot \mathrm{pr}_{23}^* g)$$

where pr_{ij} denotes the projection from $X \times Y \times Z$ to the corresponding factor. For details on correspondences see [32] or [20].

Now let us recall the definition of algebraic equivalence of algebraic cycles on a scheme X over $\mathrm{Spec}(k)$. An algebraic cycle A of dimension n on X is said to be algebraically equivalent to zero if there exists an algebraic curve C, an algebraic cycle T of dimension $n+1$ on $X \times C$, and two points a, b on C such that

$$A = T(b) - T(a)$$

where $T(a)$ is defined to be

$$\mathrm{pr}_{X*}(T.\mathrm{pr}_C^*(a))$$

for any point a in C.

With this definition of correspondence we are ready to define the notion of weak representability for Chow groups as it appears in [8]. Let X denote a smooth projective variety defined over an algebraically closed field k and let $A^i(X)$ denote the subgroup of elements of the Chow group of codimension i, which are algebraically equivalent to zero. Now let us consider an overfield K as defined in [8], that is K is an increasing union of rings R_α such that $\mathrm{Spec}(R_\alpha)$ is smooth and of finite type over $\mathrm{Spec}(k)$ and let

$$CH^i(X_K) = \varinjlim_{\alpha} CH^i(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R_\alpha))$$

and respectively

$$A^i(X_K) = \varinjlim_{\alpha} A^i(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R_\alpha)) .$$

Now we turn to the definition of weak representability. The group $A^i(X)$ is called weakly representable if there exists a smooth projective curve Γ, an algebraic cycle T supported on $\Gamma \times X$, and an algebraic subgroup B of the Jacobian variety $J(\Gamma)$ of Γ, such that the sequence

$$0 \to B(K) \to J(\Gamma)(K) \xrightarrow{T_*} A^i(X_K) \to 0$$

is exact for all $K = \bar{K} \supset k$. Here $B(K)$, $J(\Gamma)(K)$ are the groups of K points on B and $J(\Gamma)$ respectively. Since the Jacobian variety $J(\Gamma)$ is isomorphic to the group $A^1(\Gamma)$, we have a homomorphism T_* from $A^1(\Gamma_K)$ to $A^i(X_K)$ induced by the correspondence T. Also the above exact sequence means that for any $\bar{K} = K \supset k$ we have the isomorphism

$$A^i(X_K) \cong (J(\Gamma)/B)(K) .$$

Proposition 2.4.1. *Let $A^i(V)$ be weakly representable by a triple (Γ, T, B). Let W be a smooth variety and Z be a codimension i algebraic cycle supported on $W \times V$. Consider the map (after the choice of a base point $w_0 \in W$):*

$$Z_* : W \to A^i(V)$$

defined by $w \mapsto [Z(w)] - [Z(w_0)]$. Then there exists a subvariety Ω in $W \times J(\Gamma)/B$ defined over k such that

$$\Omega = \{(w, Z_*(w)) : w \in W(K)\}$$

for all $K = \bar{K} \supset k$.

Proof. See [8], Proposition 3.4. □

Proposition 2.4.2. *Let V, W be smooth projective varieties. Let $f : V \to W$ be a proper morphism that is generically finite of degree d. Then the weak representability of $A^i(V)$ implies the weak representability of $A^i(W)$.*

Proof. For the proof, see[8], Proposition 3.10. □

Proposition 2.4.3. *Let V be a smooth projective variety and let $Z \subset V$ be a smooth subvariety of V. Consider W that is obtained by blowing up V along Z. Then weak representability of $A^2(V)$ implies the weak representability of $A^2(W)$*

Proof. See [8], Proposition 3.11. □

Assuming the resolution of singularities in dimension $\leq n$ we have the following corollary of the above proposition.

Corollary 2.4.4. *Under the above assumption, the weak representability of $A^2(V)$ is a birational invariant for varieties of dimension $\leq n$.*

Proof. This follows from the previous proposition 2.4.3. □

Corollary 2.4.5. *Let $\pi : V \dashrightarrow S$ be a proper surjective rational morphism with* $\dim(V) = 3, \dim(S) = 2$. *Assume that for the geometric generic fibre we have an isomorphism*

$$V_{\overline{\eta}} \xrightarrow{\sim} \mathbb{P}^1{}_{\overline{\eta}} \ ,$$

and that $A^2(S)$ is weakly representable. Then $A^2(V)$ is also weakly representable.

Proof. See [8], Proposition 3.16. □

Weak representability of A^2 has several nice properties proven in [8].

We will also use the notion of rational weak representability, see [14]. Let V be a smooth projective variety over k. The group

$$A^i_{\mathbb{Q}}(V) = A^i(V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is said to be rationally weakly representable if there exists a smooth projective curve Γ over k and a cycle class Z in $CH^i_{\mathbb{Q}}(\Gamma \times V) = CH^i(\Gamma \times V) \otimes_{\mathbb{Z}} \mathbb{Q}$, such that for any algebraically closed field Ω containing k the homomorphism Z_* induced by z from $A^1_{\mathbb{Q}}(\Gamma_\Omega)$ to $A^i_{\mathbb{Q}}(V_\Omega)$ is surjective. In other words, rational weak representability is weak representability with coefficients in $\mathbb{Q}$.

Chapter 3

Étale monodromy

3.1 Étale site and étale sheaves

In this section we recall basics on étale sheaves and corresponding sheaf cohomology groups, and consider some examples important for what follows in the next section. Throughout the chapter all schemes will be separated.

Recall that a morphism $f : X \to Y$ is called locally finitely presented if for each x in X there are affine open neighborhoods V of $y = f(x)$ and U of x such that $f(U) \subset V$ and the $\Gamma(V, \mathscr{O}_Y)$ algebra $\Gamma(U, \mathscr{O}_X)$ is finitely presented. Let us recall the notion of a finitely presented algebra over a ring R. Let A be an R algebra, A is said to be finitely presented if A is isomorphic to a quotient of a polynomial ring in finitely many variables over R by a finitely generated ideal. A morphism $f : X \to Y$ is said to be flat if for each point $x \in X$ the local ring $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{Y,y}$ module. A locally finitely presented morphism $f : X \to Y$ is called étale if it is flat and unramified.

Étale morphisms possess the following nice properties. Open immersions are étale. If $f : X \to Y$ and $g : Y \to Z$ are étale, then so is the composite $g \circ f$. If $f : X \to X'$ and $g : Y \to Y'$ are étale S-morphisms, for a third scheme S, then so is their fibred product

$$f \times_S g : X \times_S Y \to X' \times_S Y'$$

over S. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes such that $g \circ f$ and g are étale. Then f is étale. This all can be found in Section 17.3.3 in [17].

Let X be a scheme. We consider the category Et/X of étale X-schemes. By the previous properties, Et/X has finite fiber products and X is a final object of Et/X. A family of morphisms

$$\{X_i' \xrightarrow{\phi_i} X'\}$$

is called surjective in Et/X if

$$X' = \cup_i \phi_i(X_i') \ .$$

Now we check that the set of all surjective morphisms in Et/X satisfies axioms of being covering of a Grothendieck topology (see [37] for the axioms).

Indeed, let

$$\{U_i \stackrel{\phi_i}{\to} U\}$$

be a surjective family of morphisms in Et/X and let $f : V \to U$ be a morphism in Et/X. Then we have to check that

$$\{U_i \times_U V \stackrel{\phi_i \times_U f}{\to} V\}$$

is a surjective family of morphisms. Since $\{U_i \stackrel{\phi_i}{\to} U\}$ is a surjective family we get that $\cup_i \phi_i(U_i) = U$. Then it follows by the definition of fiber product that

$$f^{-1}(U) = \cup_i(\phi_i \times_U f)(U_i \times_U V)\ .$$

But $f^{-1}(U) = V$, so we get that the first axiom of a Grothendieck topology is satisfied.

Other axioms are straightforward.

Thus, we can now define the étale site $X_{ét}$ of X by setting the category being Et/X and coverings being the set of surjective families of morphisms in Et/X. A presheaf F of sets on $X_{ét}$ is a sheaf, if for each covering $\{X'_i \to X'\}$ in $X_{ét}$ the sequence

$$F(X') \to \prod_i F(X'_i) \rightrightarrows \prod_{i,j} F(X'_i \times'_X X'_j)$$

is exact. The category of set valued sheaves on $X_{ét}$ is also called the étale topos on X. Sheaves of abelian groups are called abelian sheaves. The category of abelian sheaves on $X_{ét}$ is denoted by $\widetilde{X_{ét}}$. For each abelian sheaf F in $\widetilde{X_{ét}}$ and for each scheme X' étale over X the cohomology groups $H^q_{ét}(X', F)$ are defined and they are denoted by $H^q_{ét}(X', F)$. They can be understood as derived functors of global sections of sheaves on $X_{ét}$ and computed as Cech cohomology groups, see [37].

Let us now consider examples of étale sheaves. First of all, the étale topology is subcanonical, i.e. all representable presheaves are sheaves. By a representable sheaf we mean a representable pre-sheaf which is a sheaf. A representable pre-sheaf is a contravarinat functor from $X_{ét}$ to the category of sets, that is naturally isomorphic to $\mathrm{Hom}_X(-, A)$ for an object A in $X_{ét}$. Let G be a group scheme over X. Then we denote by G_X the sheaf on $X_{ét}$ represented by G. This is a sheaf of groups on $X_{ét}$. For each étale X-scheme X' we have $G_X(X') = \mathrm{Hom}_X(X', G)$ - the group of points of G with values in X'. If G is a commutative group scheme then G_X is an abelian sheaf in $\widetilde{X_{ét}}$.

Example 3.1.1. Let $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[t]) \times_{\mathrm{Spec}(\mathbb{Z})} X$, and for an etale scheme X' we obtain that

$$(\mathbb{G}_a)_X(X') = \mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t]) \times_{\mathrm{Spec}(\mathbb{Z})} X)$$

which is same as

$$\mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t]))$$

that is nothing but

$$\mathrm{Hom}(\mathbb{Z}[t], \Gamma(X', \mathscr{O}'_X))$$

Now we prove that the above is isomorphic to $\Gamma(X', \mathscr{O}'_X)$. Let us define

$$\Phi : \mathrm{Hom}(\mathbb{Z}[t], \Gamma(X', \mathscr{O}'_X)) \to \Gamma(X', \mathscr{O}'_X)$$

by

$$\Phi(f) = f(t)$$

on the other hand define

$$\Psi : \Gamma(X', \mathscr{O}'_X) \to \mathrm{Hom}(\mathbb{Z}[t], \Gamma(X', \mathscr{O}'_X))$$

by

$$\Psi(a) = g_a$$

where g_a is defined as follows

$$g_a(t) : t \mapsto a \ .$$

Now we prove that Φ and Ψ are inverses to each other, on one hand we have

$$\Phi(\Psi(a)) = \Phi(g_a) = g_a(t) = a$$

and on the other we have

$$\Psi(\Phi(f)) = \Psi(f(t)) = g_{f(t)}$$

but

$$g_{f(t)}(t) = f(t)$$

therefore we have

$$g_{f(t)} = f \ .$$

Therefore we get that

$$\mathrm{Hom}(\mathbb{Z}[t], \Gamma(X', \mathscr{O}'_X)) \cong \Gamma(X', \mathscr{O}'_X) \ .$$

Therefore we have

$$(\mathbb{G}_a)_X(X') \cong \Gamma(X', \mathscr{O}'_X) \ .$$

Example 3.1.2. Let us consider $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}] \times_{\mathrm{Spec}(\mathbb{Z})} X)$. Then for an etale X-scheme X' we have

$$(\mathbb{G}_m)_X(X') = \mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t, t^{-1}] \times_{\mathrm{Spec}(\mathbb{Z})} X))$$

which is same as

$$\mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]))$$

that is isomorphic to

$$\mathrm{Hom}(\mathbb{Z}[t, t^{-1}], \Gamma(X', \mathscr{O}'_X)) \ .$$

Now we show that

$$\mathrm{Hom}(\mathbb{Z}[t, t^{-1}], \Gamma(X', \mathscr{O}'_X)) \ .$$

is isomorphic to $\Gamma(X', \mathscr{O}'_X)^*$ as multiplicative groups. Let us define

$$\Phi : \mathrm{Hom}(\mathbb{Z}[t, t^{-1}], \Gamma(X', \mathscr{O}'_X)) \to \Gamma(X', \mathscr{O}'_X)^*$$

by

$$\Phi(f) = f(t)$$

since $t \cdot t^{-1} = 1$ and f is a multiplicative homomorphism from $\mathbb{Z}[t, t^{-1}]$ to $\Gamma(X', \mathscr{O}'_X)$ we have

$$f(t) \cdot f(t^{-1}) = 1$$

hence $f(t)$ belongs to $\Gamma(X', \mathscr{O}'_X)$ on the other hand define

$$\Psi : \Gamma(X', \mathscr{O}'_X)^* \to \mathrm{Hom}(\mathbb{Z}[t, t^{-1}], \Gamma(X', \mathscr{O}'_X))$$

by sending a to g_a, where $g_a(t) = a$ and $g_a(t^{-1}) = a^{-1}$. Since $g_a(t) \cdot g_a(t^{-1}) = a \cdot a^{-1} = 1$ we get that

$$g_a(t \cdot t^{-1}) = g_a(t) \cdot g_a(t^{-1}) \ .$$

So g_a is a multiplicative homomorphism from $\mathrm{Hom}(\mathrm{Spec}(\mathbb{Z}[t, t^{-1}]), \Gamma(X', \mathscr{O}_{X'}))$. As in the previous example Ψ and Φ are inverses of each other. So we get that

$$(\mathbb{G}_m)_X(X') \cong \Gamma(X', \mathscr{O}'_X)^* \ .$$

Example 3.1.3. Let us consider $\mu_n = \mathrm{Spec}(\mathbb{Z}[t]/(t^n - 1) \times_{\mathrm{Spec}\mathbb{Z}} X)$ and for an étale X-scheme X' we have

$$(\mu_n)_X(X') = \mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t]/(t^n - 1) \times_{\mathrm{Spec}\mathbb{Z}} X))$$

that is same as

$$\mathrm{Hom}_X(X', \mathrm{Spec}(\mathbb{Z}[t]/(t^n - 1)))$$

which is isomorphic to

$$\mathrm{Hom}(\mathbb{Z}[t]/(t^n - 1), \Gamma(X', \mathscr{O}'_X))$$

we claim that this isomorphic to

$$A = \{s \in \Gamma(X', \mathscr{O}'_X) : s^n = 1\} \ .$$

Let us define the map

$$\Phi : \mathrm{Hom}(\mathbb{Z}[t]/(t^n - 1), \Gamma(X', \mathscr{O}'_X)) \to \Gamma(X', \mathscr{O}'_X)$$

by

$$f \mapsto f(t)$$

since $t^n = 1$ in $\mathbb{Z}[t]/(t^n - 1)$ we have $f(t)^n = 1$ therefore $f(t)$ belongs to A. Let $a \in A$ be given then a defines the homomorphism from

$$\mathbb{Z}[t] \to \Gamma(X', \mathscr{O}'_X)$$

by the rule

$$t \mapsto a$$

and extending it to whole of $\mathbb{Z}[t]$ such that it is a homomorphism. Since $a^n = 1$, this homomorphism gives rise to a unique homomorphism

$$g_a : \mathbb{Z}[t]/(t^n - 1) \to \Gamma(X', \mathscr{O}_{X'})$$

and define

$$\Psi : A \to \mathrm{Hom}(\mathbb{Z}[t]/(t^n - 1), \Gamma(X', \mathscr{O}_{X'}))$$

by

$$\Psi(a) = g_a$$

as in the previous two examples we can show that Φ, Ψ are inverses to each other. Therefore we get that,

$$(\mu_n)_X(X') \cong \{s \in \Gamma(X', \mathscr{O}'_X) : s^n = 1\} .$$

For each natural number n we have the following exact sequence of abelian sheaves on $X_{ét}$:

$$0 \to (\mu_n)_X \to (\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X$$

where

$$(\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X$$

denotes the n-th power morphism

$$s \mapsto s^n .$$

Let A be a discrete abelian group. We denote by A_X the sheaf which is associated to the presheaf sending X' to A, for an étale X-schemes X'. Then A_X is called the constant sheaf with values in A. We have that $A_X(X')$ is the coproduct of the copies of A over the connected components of X', and it is equal to $\mathrm{Hom}_X(X', \coprod_A X)$. This means that the constant sheaf A_X is represented by the étale group scheme $\coprod_A X$ with the group structure induced by A. If F is an arbitrary sheaf on $X_{ét}$ then for each morphism $A \to F(X)$ induces a unique natural morphism of sheaves $A_X \to F$, by the universal property of a sheaf associated to a presheaf. This gives an isomorphism between $\mathrm{Hom}(A, F(X))$ and $\mathrm{Hom}(A_X, F)$.

Now, consider the constant sheaf $(\mathbb{Z}/n\mathbb{Z})_X$ on $X_{\acute{e}t}$. The morphisms $(\mathbb{Z}/n\mathbb{Z})_X$ to an abelian sheaf F, correspond to the morphisms $\mathbb{Z}/n\mathbb{Z}$ to $F(X)$, which correspond uniquely to those sections of F over X which are annihilated by n. Therefore the isomorphisms $(\mathbb{Z}/n\mathbb{Z})_X \cong (\mu_n)_X$ corresponds uniquely to the primitive n-th roots of unity on X, hence to those sections of $(\mu_n)_X$ over X which have order precisely n on each connected component of X. In particular we see that $(\mu_n)_X$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})_X$ if there exists at least one primitive n-th root of unity on X.

Next, there is a canonical isomorphism

$$H^1_{\acute{e}t}(X, (\mathbb{G}_m)_X) \cong \mathrm{Pic}(X)$$

where $\mathrm{Pic}(X)$ denotes the Picard group of X, that is the group of isomorphism classes of all invertible sheaves on X, see [37]. For any natural number n we have the n-th power morphism

$$(\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X$$

given by

$$s \mapsto s^n$$

for s in $\Gamma(X, \mathscr{O}_X)^*$. If n is invertible on X, meaning that n is prime to characteristic of $k(x)$ for all x in X, then there is an exact sequence

$$0 \to (\mu_n)_X \to (\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X \to 0 \ ,$$

which yields the following long exact sequence.

$$0 \to H^0_{\acute{e}t}(X, (\mu_n)_X) \to H^0(X, \mathscr{O}_X^*) \to H^0(X, \mathscr{O}_X^*)$$
$$\to H^1_{\acute{e}t}(X, (\mu_n)_X) \to \mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X) \to \cdots$$

It gives the exact sequence

$$0 \to H^0(X, \mathscr{O}_X^*)/nH^0(X, \mathscr{O}_X^*) \to H^1_{\acute{e}t}(X, (\mu_n)_X) \to$$
$$\ker(\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)) \to 0 \ .$$

If X is a spectrum of a local ring A and n is invertible in A then we have

$$H^1_{\acute{e}t}(X, (\mu_n)_X) \cong A^*/A^{*n} \ ,$$

because the kernel of the multiplication by n on $\mathrm{Pic}(X)$ is zero in this case.

Let X be reduced proper scheme over a separable closed field k and let characteristic of k is prime to n. Then we have the well-known isomorphism

$$H^1_{\acute{e}t}(X, (\mu_n)_X) \cong \ker(\mathrm{Pic}(X) \xrightarrow{n} \mathrm{Pic}(X)) \ ,$$

see [37].

3.2 *l*-adic étale cohomology groups

Now we are ready to introduce l-adic étale cohomology and show that it is a Weil cohomology theory. Recall that a Weil cohomology theory with coefficients in a characteristic zero field K is given by the following data. First of all, we have a contravariant functor $X \to H^*(X) = \oplus_i H^i(X)$ from the category of nonsingular (connected) projective varieties over $\mathrm{Spec}(k)$ to graded commutative K-algebras. The product of α and β in $H^*(X)$ is denoted by $\alpha \cup \beta$. For every nonsingular projective X there must exist a linear trace map $\mathrm{Tr} = \mathrm{Tr}_X : H^{2\dim(X)}(X) \to K$. For every closed irreducible subvariety Z of codimension c in X, there is given a cohomology class $\mathrm{cl}(Z)$ in $H^{2c}(X)$. The above data should satisfy the following set of 8 axioms.

(A1) For every non-singular, connected, projective variety X, all $H^i(X)$ have finite dimension over K. Furthermore, $H^i(X) = 0$ unless $0 \leq i \leq 2\dim(X)$.

(A2) (Künneth property) If X and Y are non-singular, connected, projective algebraic varieties and

$$p_X : X \times Y \to X$$

$$p_Y : X \times Y \to Y$$

are the canonical projections then the K-algebra homomorphism

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y)$$

given by

$$\alpha \otimes \beta \mapsto p_X^*(\alpha) \cup p_Y^*(\beta)$$

is an isomorphism.

(A3) (Poincaré duality) For every non-singular, connected projective algebraic variety X the trace map $\mathrm{Tr} : H^{2\dim(X)}(X) \to K$ is an isomorphism and for every i between 0 and $2\dim(X)$ the bilinear map

$$H^i(X) \otimes_K H^{2\dim(X)-i}(X) \to K$$

given by

$$\alpha \otimes \beta \mapsto \mathrm{Tr}_X(\alpha \cup \beta)$$

is a perfect pairing, that is the linear map from $H^i(X)$ to $\mathrm{Hom}_K(H^{2\dim(X)-i}(X), K)$ induced by the above bilinear map is an isomorphism.

(A4) (Trace maps and products) For every non-singular, connected, projective varieties X, Y we have

$$\mathrm{Tr}_{X\times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) = \mathrm{Tr}_X(\alpha)\mathrm{Tr}_Y(\beta)$$

for all α in $H^{2\dim(X)}(X)$ and β in $H^{2\dim(Y)}(Y)$.

(A5) (Exterior product of cohomology classes) For every non-singular, connected projective varieties X, Y and every closed irreducible subvarieties $Z \subset X$ and $W \subset Y$ we have

$$\mathrm{cl}(Z \times W) = p_X^*(\mathrm{cl}(Z)) \cup p_Y^*(\mathrm{cl}(W)) \ .$$

Here we assume that $Z \times W$ is a closed subscheme of $X \times Y$.

(A6) (Push forward of cohomology classes) For every morphism $f : X \to Y$ of non-singular, connected, projective varieties and for every irreducible closed subvariety Z of X we have for every α in $H^{2\dim(Z)}(Y)$

$$\mathrm{Tr}_X(\mathrm{cl}(Z) \cup f^*(\alpha)) = \deg(Z/f(Z)) \cdot \mathrm{Tr}_Y(\mathrm{cl}(f(Z)) \cup \alpha) \ .$$

(A7) (Pull-back of cohomology classes) Let $f : X \to Y$ be a morphism of non-singular, connected, projective varieties and $Z \subset Y$ an irreducible closed subvariety that satisfies the following considitions.

(a) All irreducible components $W_1, \cdots, W_r$ of $f^{-1}(Z)$ have pure codimenison $\dim(Z) + \dim(X) - \dim(Y)$.

(b) Either f is flat in a neighborhood of Z or Z is generically transverse to f in the sense that $f^{-1}(Z)$ is generically smooth. Here $f^{-1}(Z)$ is the scheme theoretic inverse image of Z under the morphism f. That is consider the fibred product of Z and X over Y.

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

Under these above assumptions if

$$[f^{-1}(Z)] = \sum_{i=1}^{r} m_i W_i$$

where $[f^{-1}(Z)]$ denote the fundamental cycle of $f^{-1}(Z)$, then

$$f^*(\mathrm{cl}(Z)) = \sum_{i=1}^{r} m_i \mathrm{cl}(W_i) \ .$$

(A8) (Case of a point) If $x = \mathrm{Spec}(k)$ then $\mathrm{cl}(x) = 1$ and $\mathrm{Tr}_x(1) = 1$.

For further details on Weil cohomology theories and for a discussion on the famous standard conjectures see [21].

A basic example of a Weil cohomology theory is given by singular cohomology when $k = \mathbb{C}$, when we may take $K = \mathbb{Q}$. In our thesis we will be working over an arbitrary (uncountable) ground field k and also over function fields of varieties over k, so that singular cohomology is not suitable. It was a discovery of Grothendieck that l-adic étale cohomology groups, which we are going to introduce right now, form a Weil cohomology theory with coefficients in $\mathbb{Q}_l$ for varieties over an arbitrary ground field k.

So, let k be an algebraically closed field and let l be a prime that is different from the characteristic of k. Now consider the étale topology on $\mathrm{Spec}(k)$. Then consider X to be a scheme étale over $\mathrm{Spec}(k)$. Let $\mathbb{Z}/l^m\mathbb{Z}$ denote the constant sheaf given by the ring $\mathbb{Z}/l^m\mathbb{Z}$. Consider i-th cohomology group $H^i_{\acute{e}t}(X, \mathbb{Z}/l^m\mathbb{Z})$ with values in the constant sheaf $\mathbb{Z}/l^m\mathbb{Z}$. Now we have natural homomorphism from $\mathbb{Z}/l^{m+1}\mathbb{Z}$ to $\mathbb{Z}/l^m\mathbb{Z}$. This gives us a morphism of constant sheaves, from $\mathbb{Z}/l^{m+1}\mathbb{Z}$ to $\mathbb{Z}/l^m\mathbb{Z}$ and hence we obtain a morphism

$$H^i_{\acute{e}t}(X, \mathbb{Z}/l^{m+1}\mathbb{Z}) \to H^i_{\acute{e}t}(X, \mathbb{Z}/l^m\mathbb{Z})$$

so the groups $\{H^i_{\acute{e}t}(X, \mathbb{Z}/l^m\mathbb{Z})\}_m$ forms an inverse system and we put

$$H^i_{\acute{e}t}(X, \mathbb{Z}_l) := \varprojlim_m H^i_{\acute{e}t}(X, \mathbb{Z}/l^m\mathbb{Z}) ,$$

where $\mathbb{Z}_l$ is the ring of l-adic integers. Since for each m the cohomology group $H^i_{\acute{e}t}(X, \mathbb{Z}/l^m\mathbb{Z})$ is a $\mathbb{Z}/l^m\mathbb{Z}$ module we have that, $H^i_{\acute{e}t}(X, \mathbb{Z}_l)$ carries a natural structure of a $\mathbb{Z}_l$ module. Now one defines

$$H^i_{\acute{e}t}(X, \mathbb{Q}_l) = H^i_{\acute{e}t}(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l .$$

It is an important point to note that taking cohomology does not commute with inverse limits, therefore $H^i_{\acute{e}t}(X, \mathbb{Z}_l)$ is not the étale cohomology group of X with coefficients in the constant sheaf $\mathbb{Z}_l$. It follows from the fundamental theorems on étale cohomology that when we take the l-adic cohomology groups of smooth, connected, projective varieties over k, one gets a Weil cohomology theory with coefficients in $\mathbb{Q}_l$, for details see [21].

Now recall that a étale sheaf $\mathscr{F}$ on a scheme X is called locally constant when there is an étale covering $U_i \to X$ such that $\mathscr{F}|_{U_i}$ is constant. An étale sheaf $\mathscr{F}$ on a scheme X is said to be constructible if we can write X as a union of finitely many locally closed subschemes $Y \subset X$ such that $\mathscr{F}|_Y$ is locally constant. A projective system $(\mathscr{F}_n)_n$ of étale sheaves on X is called an l-adic sheaf if all sheaves $\mathscr{F}_n$ are constructible, $\mathscr{F}_n = 0$ for all $n < 0$, $l^{n+1}\mathscr{F}_n = 0$ for all $n \geq 0$, and

$$\mathscr{F}_{n+1} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} = \mathscr{F}_{n+1}/l^{n+1} \cong \mathscr{F}_n .$$

Let Λ_n be the ring $\mathbb{Z}/n\mathbb{Z}$. For any ring R, such that n is a unit in the ring we define $\mu_n(R)$ to be the group of n-th roots of unity in R. We define $\mu_n(R)^{\otimes r}$ to be the tensor product of $\mu_n(R)$ with itself for r-times if r is an integer greater than 0, it is equal to Λ_n if r is zero, and it is equal to

$$\mathrm{Hom}_{\Lambda_n}(\mu_n(R)^{\otimes -r}, \Lambda_n)$$

if r is an integer less than zero. Now let X be a variety over a field k whose characteristic does not divide n, we then define $\Lambda_n(r)$ to be the sheaf on $X_{\acute{e}t}$ such that

$$\Gamma(U, \Lambda_n(r)) = \mu_n(\Gamma(U, \mathscr{O}_U))^{\otimes r}$$

for all $U \to X$ étale. It is important to note that if k is algebraically closed then it contains an n-th root of 1, and then each sheaf is isomorphic to the constant sheaf Λ_n, and the choice of a primitive n-th root of unity determines isomorphisms $\Lambda_n(r) \cong \Lambda_n$. Now consider the inverse system of sheaves $\Lambda_n(r)$ where r is a non-negative integer and let us denote the inverse limit of this inverse system as $\mathbb{Z}_l(r)$. Sheafify this $\mathbb{Z}_l(r)$ and consider the tensor product with the constant sheaf given by $\mathbb{Q}_l$, that is denoted as $\mathbb{Q}_l(r)$. We can now consider the cohomology $H^i_{\acute{e}t}(X, \mathbb{Q}_l(r))$ defined by the formula

$$H^i_{\acute{e}t}(X, \mathbb{Q}_l(r)) = H^i_{\acute{e}t}(X, \mathbb{Z}_l(r)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \ .$$

It should be understood as cohomology with coefficients in $\mathbb{Q}_l(r)$.

Now let us give an outline of why the l-adic cohomology with $\mathbb{Q}_l$ coefficients is a Weil cohomology theory. The description below is borrowed from [23]. Let $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$. For complete varieties X and Y there is an isomorphism

$$\oplus_{r+s=n} H^r_{\acute{e}t}(X, \Lambda) \otimes_\Lambda H^s_{\acute{e}t}(Y, \Lambda) \cong H^n(X \times Y, \Lambda) \ ,$$

loc.cit. Passing to the inverse limit $\varprojlim_n \mathbb{Z}/l^n\mathbb{Z}$ we get the isomorphism

$$\oplus_{r+s=n} H^r_{\acute{e}t}(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} H^s_{\acute{e}t}(Y, \mathbb{Z}_l) \cong H^n(X \times Y, \mathbb{Z}_l) \ .$$

Tensoring with $\mathbb{Q}_l$, we get the Künneth formula for étale cohomology with $\mathbb{Q}_l$ coefficients.

Suppose now X is an irreducible nonsingular algebraic variety and let Z be a subvariety of X. The Gysin homomorphism is the composition

$$\Lambda = H^0_{\acute{e}t}(Z, \Lambda) \to H^{2r}_{\acute{e}t, Z}(X, \Lambda(r)) \to H^{2r}_{\acute{e}t}(X, \Lambda(r))$$

where $H^{2r}_{\acute{e}t, Z}(X, \Lambda(r))$ denotes the cohomology of the pair (X, Z) (see [23] for cohomology of pairs). Now the cycle class $\mathrm{cl}_X(Z)$ is the image of 1 under the Gysin homomorphism, when Z is non-singular.

We can also extend this definition of the cycle class map to the case when Z is singular. Let Z be an irreducible subvariety of X of codimension c and let Y

be the singular locus of Z. Now we have an exact sequence of étale cohomology groups for the triple

$$(X, X \setminus Y, X \setminus Z)$$

using this exact sequence and the fact that for any closed subvariety Z of codimension c in X, $H^r_{ét}(X, \Lambda) = 0$ for $r < 2c$ we get that there is an isomorphism

$$H^{2c}_{ét,Z}(X, \Lambda) \cong H^{2c}_{ét,Z\setminus Y}(X \setminus Y, \Lambda) \ .$$

We define $\mathrm{cl}_X(Z)$ to be the image of 1 under the composition of maps

$$\Lambda \cong H^0_{ét,Z\setminus Y}(X \setminus Y, \Lambda(c)) \cong H^{2c}_{ét,Z}(X, \Lambda(c)) \to H^{2c}_{ét}(X, \Lambda(c))$$

we extend this by linearity to a homomorphism to the free abelian group generated by the irreducible subvarieties of X.

Now let us discuss the Poincaré duality axiom. For a sheaf of Λ modules $\mathscr{F}$ let us set

$$\check{\mathscr{F}}(m) = \mathrm{Hom}(\mathscr{F}, \Lambda(m))$$

where $\Lambda(m) = \mu_n^{\otimes m}$, and μ_n is the sheaf of n-th roots of unity, that

$$\mu_n(X) = \ker(\mathscr{O}_X^* \xrightarrow{n} \mathscr{O}_X^*)$$

Now the Poincaré duality axiom requires that for any non-singular algebraic variety of dimension d over an algebraically closed field k there is a unique map Tr_X from $H^{2d}_{ét}(X, \Lambda(d))$ to Λ sending $\mathrm{cl}(P)$ to 1 for any closed point on X and it is an isomorphism, this map Tr_X is called the trace map. Moreover for any locally constant sheaf $\mathscr{F}$ of Λ modules there are canonical pairings

$$H^r_{ét}(X, \mathscr{F}) \times H^{2d-r}_{ét}(X, \check{\mathscr{F}}(d)) \to H^{2d}(X, \Lambda(d)) \cong \Lambda$$

which are perfect pairings of finite groups. If X is quasi-projective then the above cohomologies can be identified with the Cech cohomology groups and the pairing can be defined by the usual cup-product formula. Now if we use the Poincaré duality theorem for the case when $\mathscr{F}$ is the constant sheaf given by $\mathbb{Z}/l^n\mathbb{Z}$ and we consider the inverse limit of the cohomology groups with coefficients in the constant sheaf given by $\mathbb{Z}/l^n\mathbb{Z}$, we get the validity of the Poincaré duality axiom for the l-adic cohomology.

Finally, we will need to see how the Gysin map arises from Poincaré duality. Let $\pi : Y \longrightarrow X$ be a proper morphism of smooth connected separated varieties over an algebraically closed field k. Let $a = \dim(X), d = \dim(Y), e = d - a$, then there is a restriction map

$$\pi^* : H^{2d-r}_{ét}(X, \Lambda(d)) \to H^{2d-r}_{ét}(Y, \Lambda(d))$$

by the duality we get the map

$$\pi_* : H^r_{ét}(Y, \Lambda) \to H^{r-2e}(X, \Lambda(-e)) \ .$$

It is a fact that when π is a closed immersion then π_* is the Gysin map that we have discussed above. This is a consequence of the proof of the Poincaré duality. In particular for a subvariety Y inside X we have

$$\pi_*(1) = \mathrm{cl}_X(Y)$$

where 1 is the unity of $H^0_{\acute{e}t}(Y, \Lambda)$.

3.3 Étale fundamental group

The key technique used in the thesis is étale monodromy. We discuss this important notion in the remaining three sections of this chapter. We begin with some notations and terminologies. For a scheme S let $\mathscr{F}_S$ denote the category whose objects are finite, étale covers of S and morphisms are morphisms of schemes over S. Let Ω be an algebraically closed field. Let

$$\overline{s} : \mathrm{Spec}(\Omega) \to S$$

be a geometric point. For an object $X \to S$ in $\mathscr{F}_S$ we consider the geometric fiber $X \times_S \mathrm{Spec}(\Omega)$ over $\overline{s}$. We denote by $F_{\overline{s}}(X)$ the underlying set of $X \times_S \mathrm{Spec}(\Omega)$. Let $f : X \to Y$ be a morphism between two schemes in $\mathscr{F}_S$, then we get a morphism

$$X \times_S \mathrm{Spec}(\Omega) \to Y \times_S \mathrm{Spec}(\Omega)$$

therefore we get a set theoretic map

$$F_{\overline{s}}(X) \to F_{\overline{s}}(Y) .$$

We call it the fiber functor at the geometric point of $\overline{s}$. Now let us recall the notion of a natural transformation of functors from a category $\mathscr{C}_1$ to another $\mathscr{C}_2$. Let F, G be two functors from $\mathscr{C}_1$ to $\mathscr{C}_2$. Then a morphism f between F and G is a collection of morphisms in $\mathscr{C}_2$

$$f(X) : F(X) \to G(X)$$

one for each object X in $\mathscr{C}_1$ such that for any morphism

$$\phi : X \to Y$$

we have that,

$$\begin{array}{ccc} F(X) & \xrightarrow{f(X)} & G(X) \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle G(\phi)} \\ F(Y) & \xrightarrow{f(Y)} & G(Y) \end{array}$$

Now given a functor F from a category $\mathscr{C}_1$ to $\mathscr{C}_2$ an automorphism F is a morphism of functors

$$F \to F$$

that has a two sided inverse. Let $Aut(F)$ be the collection of all automorphisms of F. Now a composition of two automorphisms is again an automorphism, making $Aut(F)$ into a group, this is called the automorphism group of F. Note that for all ϕ in $Aut(F)$ and for an object C in $\mathscr{C}_1$, we have an automorphism

$$\phi(C) : F(C) \to F(C)$$

therefore if F is set valued then $Aut(F)$ acts on $F(C)$ for all C in $\mathscr{C}_1$. Let us now recall the definition of the étale fundamental group as in [15].

Given a scheme S and a geometric point

$$\overline{s} : \mathrm{Spec}(\Omega) \to S$$

we define the étale fundamental group $\pi_1(S, \overline{s})$ as the automorphism group of the fiber functor $F_{\overline{s}}$.

Since $F_{\overline{s}}$ is set valued there is a natural left action of $\pi_1(S, \overline{s})$ on $F_{\overline{s}}(X)$ for each X in $\mathscr{F}_S$.

Theorem 3.3.1. *Let S be a connected scheme and let*

$$\overline{s} : \mathrm{Spec}(\Omega) \to S$$

be a geometric point.

(1) The group $\pi_1(S, \overline{s})$ is a profinite group and its action on $F_{\overline{s}}(X)$ is continuous for every X in $\mathscr{F}_S$.

(2) The functor $F_{\overline{s}}$ defines an equivalence of categories between $\mathscr{F}_S$ and the category of left $\pi_1(S, \overline{s})$-sets.

The proof of the theorem will be given later. Let us now consider the following definition.

Let $\mathscr{C}$ be a category and F a set valued functor on $\mathscr{C}$. We say that F is pro-representable if there exists an inverse system $P = (P_\alpha, \phi_{\alpha\beta})$ of objects in $\mathscr{C}$ indexed by a directed partially ordered set Λ and a functorial isomorphism

$$\varinjlim \mathrm{Hom}(P_\alpha, X) \cong F(X) \ .$$

Since $(P_\alpha, \phi_{\alpha\beta})$ forms an inverse system we have that $\mathrm{Hom}(P_\alpha, X)$ forms a direct system indexed by the same indexing set Λ. Recall that by definition the direct limit of a direct system $(S_\alpha, \phi_{\alpha\beta})$ of sets is the disjoint union of S_α modulo the following equivalence relation, where s_α in S_α is equivalent to s_β in S_β, if and only if there exists $\gamma \geq \alpha, \beta$ such that

$$\phi_{\alpha\gamma}(s_\alpha) = \phi_{\beta\gamma}(s_\beta) \ .$$

Remark 3.3.2. If F is pro-representable by an inverse system $(P_\alpha, \phi_{\alpha\beta})$ then for each α we have the identity map of P_α which is an element of $\mathrm{Hom}(P_\alpha, P_\alpha)$, hence gives a class in

$$\varinjlim \mathrm{Hom}(P_\beta, P_\alpha) \cong F(P_\alpha) \ .$$

Proposition 3.3.3. *Let S be a connected scheme and let*

$$\overline{s} : \mathrm{Spec}(\Omega) \to S$$

be a geometric point, then the functor $F_{\overline{s}}$ is pro-representable.

Proof. Take the index set Λ to be the set of all finite étale Galois covers $P_\alpha \to S$. Define $P_\alpha \leq P_\beta$ if there is a morphism

$$P_\beta \to P_\alpha \ .$$

This partially ordered set is directed because if P_α, P_β are given then by the following lemma applied to a connected component Z of $P_\alpha \times_S P_\beta$ we obtain P_γ with maps

$$P_\gamma \to Z \to P_\alpha$$

$$P_\gamma \to Z \to P_\beta \ .$$

The lemma is as follows.

Lemma 3.3.4. *Let $\phi : X \to S$ be a connected finite étale cover. Then there is a morphism $\pi : P \to X$ such that*

$$\phi \circ \pi : P \to S$$

is a finite étale Galois cover and moreover any S-morphism from a Galois cover to S, factors through π.

Proof. For a proof see [36] proposition 5.3.9. □

Now the objects of the inverse system are P_α themselves. Now the next we have to define the morphisms $\phi_{\alpha\beta}$. Now for each P_α fix an element p_α in $F_{\overline{s}}(P_\alpha)$. Now we now that $P_\beta \to S$ is Galois therefore the automorphism group $Aut(P_\beta|S)$ acts transitively on geometric fibers. Since we have morphism $\phi : P_\beta \to P_\alpha$ we get a morphism

$$F_{\overline{s}}(\phi) : F_{\overline{s}}(P_\beta) \to F_{\overline{s}}(P_\alpha)$$

consider the set

$$F_{\overline{s}}(\phi)^{-1}(p_\alpha)$$

take an element in this inverse image say p'_α. Since $Aut(P_\beta|S)$ acts transitively on the geometric fiber there is a λ in $Aut(P_\beta|S)$ such that

$$\lambda(p_\beta) = p'_\alpha$$

this λ is unique. Because if we have two λ_1, λ_2 such that

$$\lambda_1(p_\beta) = \lambda_2(p_\beta)$$

then we have $\lambda_1 = \lambda_2$ by the following lemma.

Lemma 3.3.5. *If $Z \to S$ is a connected scheme and let ϕ_1, ϕ_2 be two S-morphisms from Z to X where X is a finite étale cover of S. Suppose that for some geometric point*

$$z : \mathrm{Spec}(\Omega) \to Z$$

we have that

$$\phi_1 \circ z = \phi_2 \circ z$$

then

$$\phi_1 = \phi_2 \ .$$

Proof. See [36] corollary 5.3.3. □

Then we have that

$$\phi(\lambda(p_\beta)) = \phi(p'_\alpha) = p_\alpha$$

and we define

$$\phi_{\alpha\beta} = \phi \circ \lambda$$

by construction of $\phi_{\alpha\beta}$ we have that

$$F_{\overline{s}}(\phi_{\alpha\beta})(p_\beta) = p_\alpha \ .$$

Now let X be in $\mathscr{F}_S$, and for every P_α in Λ there is a natural map from $\mathrm{Hom}(P_\alpha, X)$ to $F_{\overline{s}}(X)$ by sending

$$\phi \mapsto F_{\overline{s}}(\phi)(p_\alpha) \ .$$

By using the fact that

$$F_{\overline{s}}(\phi_{\alpha\beta})(p_\beta) = p_\alpha$$

we have that these maps are compatible with the transition maps $\phi_{\alpha\beta}$ in the inverse system defined by P_α's and therefore by the universality of the direct product we have a map from

$$\varinjlim \mathrm{Hom}(P_\alpha, X)$$

to $F_{\overline{s}}(X)$. Now we find the inverse to this map. To do that we can assume that X is connected otherwise we can take disjoint unions. Consider the Galois closure $\pi : P \to X$ which exists by 3.3.4. Here P is one of the P_α such that α belongs to Λ. Let x be in $F_{\overline{s}}(X)$, and consider $\pi^{-1}(x)$ inside $F_{\overline{s}}(P_\alpha)$, take x' in $\pi^{-1}(x)$. Since the automorphism group $Aut(P_\alpha|S)$ acts transitively on the geometric fiber $F_{\overline{s}}(P_\alpha)$, there exists a λ in $Aut(P_\alpha|S)$ such that

$$\lambda(p_\alpha) = x' \ .$$

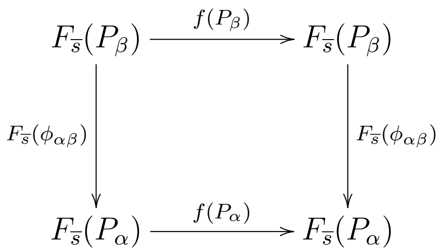
This λ is unique by 3.3.5. So that we have $\pi \circ \lambda$ sends p_α to x. Send x to the class of $\pi \circ \lambda$ in $\varinjlim \mathrm{Hom}(P_\alpha, X)$, this gives the required inverse. This finishes the proof of 3.3.3. □

Definition 3.3.6. An automorphism of the system $(P_\alpha, \phi_{\alpha\beta})$ is a collection of λ_α in $Aut(P_\alpha|S)$ such that

$$\phi_{\alpha\beta} \circ \lambda_\beta = \lambda_\alpha \circ \phi_{\alpha\beta} \ .$$

Corollary 3.3.7. *Every automorphism of the functor $F_{\overline{s}}$ comes from a unique automorphism of the inverse system $(P_\alpha, \phi_{\alpha\beta})$.*

Proof. By definition of an automorphism f of $F_{\overline{s}}$ we have that



this gives us that

$$F_{\overline{s}}(\phi_{\alpha\beta}) \circ f(P_\alpha)(p_\alpha) = f(P_\beta) \circ F_{\overline{s}}(\phi_{\alpha\beta})(p_\alpha) = f(P_\beta)(p_\beta)$$

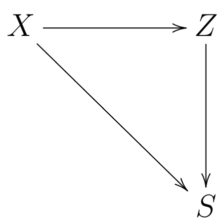
therefore we have that $(f(P_\alpha)(p_\alpha))_\alpha$ forms another system of distinguished elements as in the previous proposition. Since P_α are Galois therefore there exists a unique λ_α such that

$$\lambda_\alpha(p_\alpha) = f(P_\alpha)(p_\alpha) \ .$$

This gives by the commutativity of the above diagram an automorphism of the inverse system $(P_\alpha, \phi_{\alpha\beta})$. □

Corollary 3.3.8. *The automorphism groups $Aut(P_\alpha|S)$ form an inverse system whose inverse limit is $\pi_1(S, \overline{s})$.*

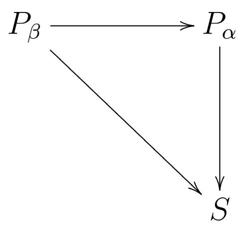
Proof. The inverse system comes from the proposition 5.3.8 in [36] which states that given a finite Galois cover $X \to S$, there is a one-to-one correspondence between subgroups of $Aut(X|S)$ and the intermediate covers Z fitting into the diagram.



Now if $P_\alpha \leq P_\beta$, then there is

$$\phi_{\alpha\beta} : P_\beta \to P_\alpha \ .$$

Since P_β is Galois over S we have the commutative diagram



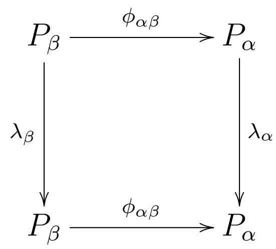
and we can realize P_α as a quotient of P_β by a subgroup H of $Aut(P_\beta|S)$. Therefore there is a surjective map

$$f_{\alpha\beta} : Aut(P_\beta|S) \to Aut(P_\alpha|S) \ .$$

Now the elements of the inverse limit are those element $(\lambda_\alpha)_\alpha$ such that

$$f_{\alpha\beta}(\lambda_\beta) = \lambda_\alpha$$

for all $\alpha \leq \beta$. This gives us the following commutative diagram



therefore the elements of the inverse limit of the system $Aut(P_\alpha|S)^{op}$ is exactly the automorphism of $F_{\overline{s}}$ by the previous corollary 3.3.7. □

Now we come to the proof of 3.3.1. First we state the following corollary from [36]. For that we need the following result.

Corollary 3.3.9. *If $\phi : X \to S$ is a connected finite étale cover, the non-trivial elements of $Aut(X|S)$ act without fixed points on each geometric fibers, hence $Aut(X|S)$ is finite.*

Proof. For a proof please see [36] corollary 5.3.4. □

Now we can give the proof for 3.3.1:

Proof. The group $\pi_1(S, \overline{s})$ is the inverse limit of the automorphism groups $Aut(P_\alpha|S)$ which are finite, therefore the inverse limit is a profinite group.

Now we come to the proof of the second statement. We first prove the essential surjectivity. Let E be a finite continuous left $\pi_1(S, \overline{s})$ set. Without loss of generality we can assume that $\pi_1(S, \overline{s})$ acts transitively on E. The stabiliser of x in E, is H_x and we have by the orbit stabiliser theorem

$$[\pi_1(S, \overline{s}) : H_x] \cong E$$

therefore H_x is of finite index in $\pi_1(S, \overline{s})$. Since H_x is close and of finite index in $\pi_1(S, \overline{s})$ we have that H_x is open and therefore contains a open normal V_α subgroup which is the kernel of the homomorphism

$$\pi_1(S, \overline{s}) \to Aut(P_\alpha|S)$$

since V_α's form a basis of open neighborhoods of 1 in $\pi_1(S, \overline{s})$. Let H be the image of H_x inside $Aut(P_\alpha|S)$. Let us consider the the quotient $X = P_\alpha/H$ now we have to prove that $F_{\overline{s}}(P_\alpha/H)$ is isomorphic to E. Now consider $Aut(P_\alpha|S)$, then we prove that

$$Aut(P_\alpha|S)/H \cong F_{\overline{s}}(P_\alpha/H) \ .$$

Consider the distinguished element p_α in $F_{\overline{s}}(P_\alpha)$, then consider the image $\overline{p_\alpha}$ under the map

$$F_{\overline{s}}(P_\alpha) \to F_{\overline{s}}(P_\alpha/H) \ .$$

Now let λ be an element in $Aut(P_\alpha|S)$, then that gives rise to an element $\overline{\lambda}$ in $Aut(P_\alpha/H|S)$. Consider the homomorphism

$$\Phi : Aut(P_\alpha|S) \to F_{\overline{s}}(P_\alpha/H)$$

given by

$$\Phi(\lambda) = \overline{\lambda}(\overline{p_\alpha})$$

since $P_\alpha \to S$ is Galois we have that Φ is surjective, and since H acts trivially on the geometric fibers of P_α/H, we get that

$$F_{\overline{s}}(P_\alpha/H) \cong Aut(P_\alpha|S)/H$$

and we have

$$Aut(P_\alpha|S)/H \cong \pi_1(S, \overline{s})/H_x \cong E \ .$$

Therefore the functor $F_{\overline{s}}$ is essentially surjective. Now we have to prove that it is fully faithful. For that let us consider $\mathrm{Hom}(F_{\overline{s}}(X), F_{\overline{s}}(Y))$ we have to prove that the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F_{\overline{s}}(X), F_{\overline{s}}(Y))$$

is injective and surjective. Since a map $f : F_{\overline{s}}(X) \to F_{\overline{s}}(Y)$ is $\pi_1(S, \overline{s})$ equivariant we have that

$$f(gx) = gf(x)$$

and therefore the stabiliser of x in $F_{\bar{s}}(X)$ is contained in the stabiliser of $f(x)$. Now consider the Galois closure P of X and let $H_x, H_{f(x)}$ correspond to the subgroups those are images of the stabiliser of $x, f(x)$ in $\pi_1(S, \bar{s})$. Then we can realise X and Y as $P_\alpha/H_x, P_\alpha/H_{f(x)}$, since $H_x \subset H_{f(x)}$ we get a map from $X \to Y$. This gives the surjectivity of

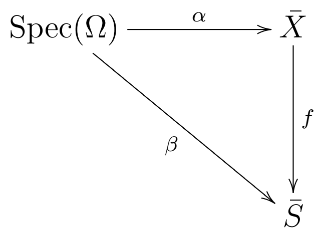
$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F_{\bar{s}}(X), F_{\bar{s}}(Y))$$

and the injectivity follows from 3.3.5. □

3.4 The tame fundamental group of Abhyankar

In this section we discuss the important notion of the tame fundamental group, which will be used in the étale monodromy in the next section. Tame fundalmental groups were introduced by Abhyankar, see [1] and [2].

Consider a finite surjective morphism from a normal connected scheme $\bar{X}$ to a normal connected scheme $\bar{S}$. Let α be a geometric point of $\bar{X}$, that is a morphism from the spectrum of a separably closed field Ω to $\bar{X}$ and β be its image point in $\bar{S}$, that is we have the commutative diagram.



Let a be a point of $\bar{X}$ associated with α and $b = f(a)$. Assume that the local rings of the structure sheaf at a, b satisfy

$$\dim(\mathscr{O}_{\bar{X},a}) = \dim(\mathscr{O}_{\bar{S},b}) = 1 \ .$$

This assumption geometrically means the following. Let a belong to an affine open neighborhood Spec(A) and let a be given by the prime ideal $\mathfrak{p}$. Then $\mathscr{O}_{\bar{X},a}$ is the local ring $A_{\mathfrak{p}}$ and

$$\dim(\mathscr{O}_{\bar{X},a}) = \dim(A_{\mathfrak{p}}) = 1$$

this means that the prime ideal $\mathfrak{p}$ is of height one, that is the codimension of a in $\bar{X}$ is 1. So the point a in $\bar{X}$ determines a divisor on $\bar{X}$. Similarly b determines a divisor on $\bar{S}$. Let $\bar{X}$ be Galois over $\bar{S}$, that is if $G = Aut(\bar{X}/\bar{S})$ be the automorphism group of $\bar{X}$ over $\bar{S}$, and $F(\bar{X})$, $F(\bar{S})$ are the field of rational functions of $\bar{X}, \bar{S}$ then we have

$$F(\bar{X})^G = F(\bar{S}) \ .$$

If $\bar{X}$ is étale over $\bar{S}$, then it means that $\bar{X}$ is a Galois covering of $\bar{S}$. Let e be the order of the ramification group

$$G_\alpha = \{\sigma \in G | \sigma \circ \alpha = \alpha\}$$

and assume e is invertible in $\mathscr{O}_{\bar{X},a}$. We then say that $\bar{X}$ is tamely ramified at the point α.

Now let A be a closed subscheme of pure codimension 1 in $\bar{S}$, S the complement of A in $\bar{S}$ and

$$X \to S$$

is a Galois covering space of S with automorphism group $Aut(X|S) = G'$. Then we claim that there exists a rational map from X to a uniquely determined normal irreducible scheme $\bar{X}$ over $\bar{S}$, that is finite over $\bar{S}$ on which G' acts. Here the scheme $\bar{X}$ is the normalization of $\bar{S}$ in the function field $F(X)$ of X, on the other hand the scheme denoted by $\bar{X}$ at the beginning of this section was already a normal connected scheme. We prove the above assertion about the existence of a rational map from X to $\bar{X}$ in details. Let us do everything at the level of affine integral schemes over a field k. Let

$$\bar{S} = \mathrm{Spec}(R), \quad X = \mathrm{Spec}(B)$$

Since

$$S = \bar{S} \smallsetminus A$$

and A is of pure codimension 1 closed subscheme, we can cover S by $D(f)$, where $f \in R$. For simplicity let us assume that $S = \mathrm{Spec}(R_f)$. Then we have an embedding

$$R \hookrightarrow R_f$$

corresponding to the morphism

$$S \to \bar{S}\ .$$

Now the morphism $X \to S$ is finite and surjective, that induces an embedding

$$R_f \hookrightarrow B$$

and moreover B is embedded in its field of fractions $B_{(0)}$. Let us consider the integral closure of R inside $B_{(0)}$ and call it $\widetilde{R}$. Then we have a morphism

$$R \to \widetilde{R} \hookrightarrow B_{(0)}\ .$$

At the level of schemes we have the following from the above analysis.

$$\eta \hookrightarrow X \to S \hookrightarrow \bar{S}$$

where η is the generic point of X. Also we have a morphism

$$\bar{X} \to \bar{S}$$

and

$$\eta \to \bar{X} \ .$$

This map $\eta \to \bar{X}$ defines a rational morphism from X to $\bar{X}$. Since $\bar{X}$ is the normalization of $\bar{S}$ in the function field $F(X)$ we have by definition of a normalization, the morphism $\bar{X} \to \bar{S}$ is finite. We call the covering space $X \to S$ to be tamely ramified over S (with respect to the embedding $S \hookrightarrow \bar{S}$) when $\bar{X}$ is tamely ramified at all geometric points that lie over a generic point of an irreducible component of A. Let

$$s : \mathrm{Spec}(K) \to S$$

be a geometric point of S. We consider the category of those pointed Galois coverings spaces

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\ t\ } & X \\ & \searrow^{s} & \downarrow \\ & & S \end{array}$$

over (S, s) with Galois group $Aut(X/S)$ that are tamely ramified. One can show by using Abhyankar's lemma A.I.11 in [12] that this category is filtered. Proceeding as in 3.3.8 and using Abhyankar's lemma we can show that $Aut(X/S)$ forms an inverse system, where we take the inverse system over all tamely ramified pointed Galois covering spaces of (S, s). Now consider the inverse limit

$$\varprojlim_{(X,t)} Aut(X/S) \ ; ,$$

where the limit is taken over all tamely ramified pointed covering spaces of (S, s). We call it the tame fundamental group of (S, s) with respect to the embedding $S \hookrightarrow \bar{S}$ and denote it as $\pi_1^{\mathrm{tame}}(S, s)$. Note that this definition of the tame fundamental group does depend on the choice of A. It is a factor group of the full fundamental group $\pi_1(S, s)$. The kernel of this homomorphism is the subgroup in $\pi_1(S, s)$ such that the corresponding morphism $\bar{X} \to \bar{S}$ is not tamely ramified.

Let n be a natural number such that all primes p not dividing n are invertible in $\mathscr{O}_{\bar{S},b}$. Let

$$\hat{\mathbb{Z}}^{(n)}(1) = \varprojlim_{(e,n)=1} \mu_e(\Omega)$$

here the system of all the $\mu_e(\Omega)$ is considered as a projective system with the mapping

$$\mu_e(\Omega) \to \mu_{e'}(\Omega)$$

$$\xi \mapsto \xi^{e/e'}$$

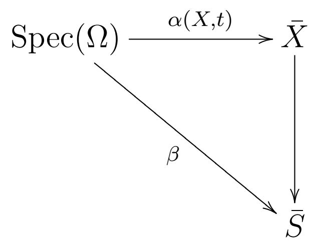
for e' dividing e. Now consider the following directed system of pointed schemes over $\bar{S}$. We say that $(X,t) \leq (X',t')$, if there exists a morphism $f : \bar{X}' \to \bar{X}$ such that $f(t) = t'$. This gives us a morphism from $\mathrm{Hom}_{\bar{S}}(\mathrm{Spec}(\Omega), \bar{X}')$ to $\mathrm{Hom}_{\bar{S}}(\mathrm{Spec}(\Omega), \bar{X})$ over a generic geometric point

$$\beta : \mathrm{Spec}(\Omega) \to A \subset S \ .$$

Now we consider the inverse limit of this system $\varinjlim_{(X,t)} \mathrm{Hom}_{S}(\mathrm{Spec}(\Omega), \bar{X})$ of pointed Galois covering spaces over a generic geometric point

$$\beta : \mathrm{Spec}(\Omega) \to A \subset S \ .$$

That is a coherent system of geometric points



by lemma 12 in [12] we have an isomorphism

$$\mu_e(\Omega) \cong G_\alpha \subset G$$

where e is the order of the ramification group

$$G_\alpha = \{\sigma | \sigma \circ \alpha = \alpha\} \ .$$

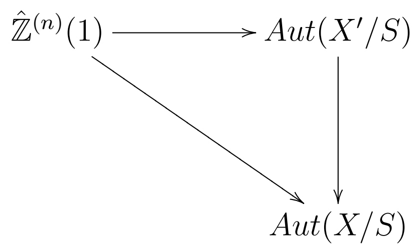
By this lemma we get an isomorphism

$$\mu_e(\Omega) \to G_{\alpha(X,t)} \subset Aut(X/S)$$

where $X \to S$ is tamely ramified. This gives us a morphism

$$\phi^{\alpha(X,t)} : \hat{\mathbb{Z}}^{(n)}(1) \to Aut(X/S) \ .$$

Now we prove that the diagram



is commutative. We have following commutative diagram at the level of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & \searrow & \downarrow \\ & & S \end{array}$$

This gives us the commutative diagram

$$\begin{array}{ccc} \mu_e(\Omega) & \longrightarrow & G_{\alpha'} \\ & \searrow & \downarrow \\ & & G_{\alpha} \end{array}$$

where α, α' are the geometric points such that we have the following.

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \xrightarrow{\alpha'} & X' \\ & {\scriptstyle\alpha}\searrow & \downarrow \\ & & X \end{array}$$

This gives us that the diagram is commutative.

$$\begin{array}{ccc} \hat{\mathbb{Z}}^{(n)}(1) & \longrightarrow & Aut(X'/S) \\ & \searrow & \downarrow \\ & & Aut(X/S) \end{array}$$

Therefore by the universality of the inverse limit we have a homomorphism

$$\phi^{\alpha} : \hat{\mathbb{Z}}^{(n)}(1) \to \pi_1^{\text{tame}}(S, s) \ .$$

If we choose some other element α' from the inverse limit $\varinjlim_{(X,t)} \mathrm{Hom}_{\bar{S}}(\mathrm{Spec}(\Omega), \bar{X})$ then $\phi^{\alpha'}$ arises by conjugation by an element of $\pi_1^{\text{tame}}(S, s)$. This is because of the following. α, α' are given by a coherent system of geometric points $\alpha(X, t), \alpha'(X, t)$ such that we have

$$\phi^{\alpha'(X,t)}(\xi) = \sigma_X \phi^{\alpha(X,t)}(\xi) \sigma_X^{-1}$$

for σ_X in $Aut(X/S)$ and for ξ in $\mu_e(\Omega)$. Therefore passing to the inverse limit we get a σ in $\pi_1^{\text{tame}}(S, s)$ such that

$$\phi^{\alpha'}(\xi) = \sigma \phi^{\alpha}(\xi) \sigma^{-1}$$

for ξ in $\hat{\mathbb{Z}}^{(n)}(1)$. Therefore the conjugacy class of ϕ^α only depend on $\beta : \mathrm{Spec}(\Omega) \to \bar{S}$.

Definition 3.4.1. We denote by

$$\gamma_\beta : \hat{\mathbb{Z}}^{(n)}(1) \to \pi_1^{\mathrm{tame}}(S, s)$$

any element from the conjugacy class of homomorphisms

$$\phi^\alpha : \hat{\mathbb{Z}}^{(n)}(1) \to \pi_1^{\mathrm{tame}}(S, s) \ .$$

Now let $\mathbb{P}^1$ be the projective line over the separably closed base field k and K is a separably closed extension field. We consider the geometric points

$$s : \mathrm{Spec}(K) \longrightarrow \mathbb{P}^1$$

and

$$\beta_i : \mathrm{Spec}(K) \longrightarrow \mathbb{P}^1 \ ; ,$$

$i = 0, \cdots, r$. Let a, b_i for $i = 0, \cdots, r$ be points associated to the geometric points s, β_i for $i = 0, \cdots, r$ respectively, and b_i's are pairwise distinct and the set

$$A = \{b_0, \cdots, b_r\}$$

is closed. Then we want to determine the tame fundamental group

$$\pi_1^{\mathrm{tame}}(\mathbb{P}^1 \setminus A, s)$$

of the pointed projective line. Now we state without proof the following two results which are going to be used in the next section.

Proposition 3.4.2. *For a suitable choice of the homomorphism*

$$\gamma_\beta : \hat{\mathbb{Z}}^{(p)}(1) \to \pi_1^{\mathrm{tame}}(\mathbb{P}^1 \setminus A, s)$$

in their conjugacy classes the images generate a dense subgroup in $\pi_1^{\mathrm{tame}}(\mathbb{P}^1 \setminus A, s)$. Here p is one if the characteristic of the field is 0 or p if the characteristic is p.

Proof. For a proof please see [12] A.I.15. □

Now consider the n-dimensional projective space $\mathbb{P}^n$ over an algebraically closed field k, an irreducible reduced hypersurface F in $\mathbb{P}^n$ and a line D in $\mathbb{P}^n$ that meets F only transversally at smooth points. Let $F \cap D$ is $\{b_0, \cdots, b_r\}$ and let

$$\beta_\nu : \mathrm{Spec}(k) \to D$$

for $\nu = 0, \cdots, r$ be the geometric points associated with the underlying ordinary points b_ν and let

$$s : \mathrm{Spec}(K) \to D \setminus A \subset \mathbb{P}^n \setminus F$$

be another geometric point of $D \setminus A$ and thus also of $\mathbb{P}^n \setminus F$. Here K is a separably closed extension field of k. Then we have the following result.

Proposition 3.4.3. *There is a homomorphism*

$$q : \pi_1^{\mathrm{tame}}(D \setminus A, s) \to \pi_1^{\mathrm{tame}}(\mathbb{P}^n \setminus F, s)$$

which is surjective. The homomorphisms

$$q \circ \gamma_\beta : \hat{\mathbb{Z}}^{(p)}(1) \to \pi_1^{\mathrm{tame}}(\mathbb{P}^n \setminus F, s)$$

are conjugate in $\pi_1(\mathbb{P}^n \setminus F, s)$.

Proof. For an outline of the proof please see the discussion after A.I.16 in [12]. □

3.5 Étale Monodromy and the Picard-Lefschetz formula

In this section, we consider an even dimensional smooth irreducible projective variety and consider a Lefschetz pencil on it and study the action of the étale fundamental group on the cohomology of the fibers of the Lefschetz pencil.

So let X be an irreducible smooth projective variety over an algebraically closed ground field k of arbitrary characteristic and fix an embedding of X in $\mathbb{P}^N$ such that the embedding is a Lefschetz embedding [12] [chapter III, definition 1.4]. By Lefschetz embedding we mean the following. Let X be embedded inside some $\mathbb{P}^N$, then this embedding is called a Lefschetz embedding if there exists a closed set A in $(\mathbb{P}^N)^\vee$ of codimension greater or equal to 2 such that the following conditions are satisfied by the hyperplanes H with co-ordinates in k, that does not lie in A.

(I) H does not contain any connected component of X.

(II) The scheme theoretic intersection $H \cap X$ contains at most one singular point and when such a point exists it is an ordinary double point.

Existence of a Lefschetz embedding in arbitrary characteristic and over an infinite field is guaranteed by the following proposition according to [12] proposition 1.5 in chapter III, which was originally proved in SGA 7, II, exp. XVII.

Proposition 3.5.1. *Let X be an irreducible smooth projective variety,* $\dim X \geq 1$ *over an infinite base field k. Let $j : X \hookrightarrow \mathbb{P}^r$ be a projective embedding. Denote by*

$$j(d) : X \hookrightarrow \mathbb{P}^N$$

the composition of j with the Veronese embedding $\mathbb{P}^r \to \mathbb{P}^N$ of degree d. Then:

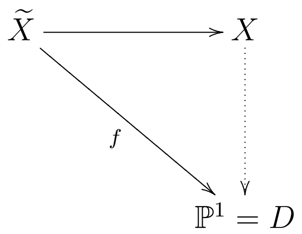
(a) In the case of characteristic 0, $j = j(1)$ *is a Lefschetz embedding.*

(b) $j(d)$ *is always a Lefschetz embedding for* $d \geq 2$.

Now let the dimension of X be even and

$$\dim(X) = n + 1$$

where $n = 2m + 1$. Let F be the dual variety of X, that is the collection of all the hyperplanes $H \subset \mathbb{P}^N$ that touches X at some point. We need Lefschetz embedding because we want to consider Lefschetz pencils on the projective variety X and the existence of a Lefschtez pencil is guaranteed by the existence of a Lefschetz embedding. Let us consider a Lefschetz pencil on X, that is, take projective line D in $\mathbb{P}^N$ such that it gives rise to a rational morphism from X to D. Resolving the indeterminacy locus of this rational morphism we get the following diagram,



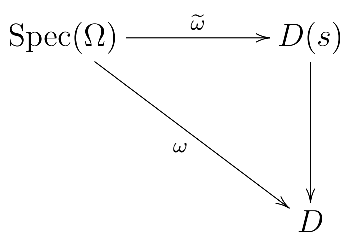
where $\widetilde{X}$ is the blow-up of X along the indeterminacy of the rational morphism

$$X \dashrightarrow \mathbb{P}^1 = D\ .$$

So we get a regular morphism from $\widetilde{X}$ to D, by definition of a Lefschetz pencil we have that, the fibers of this morphism is either smooth or it contains exactly one ordinary double point. Let F meet D transversally at only at smooth points, such that for any point $s \in F \cap D = A$ we have $f^{-1}(s)$ is singular and it contains exactly one singular point, which is an ordinary double point. Let Ω be the separable closure of the function field $k(D) = K$ and let

$$\omega : \operatorname{Spec}(\Omega) \to D \smallsetminus A$$

be the corresponding geometric point. For a point s in A let $R(s)$ be the strict Henselization of the ring $\mathscr{O}_{D,s}$ and is denoted by $\widetilde{\mathscr{O}}_{D,s}$ and let $D(s) = \operatorname{Spec}(\widetilde{\mathscr{O}}_{D,s})$. Then since we have a homomorphism $\phi : R(s) \to \Omega$ that identifies Ω with the separable closure of the quotient field K of $R(s)$, we get the following commutative diagram.



We consider the following map

$$\operatorname{Gal}(\Omega/K) \to \hat{\mathbb{Z}}^{(p)}(1) = \varprojlim_{p \nmid n} \mu_n(k)$$

that is induced by the system of homomorphisms

$$\Phi_r : \mathrm{Gal}(\Omega/K) \to \mu_n(k)$$

given by

$$\Phi_r(\sigma) = \sigma(\phi(\pi)^{1/r})/\phi(\pi)^{1/r}$$

by the universality of the inverse limit we have a homomorphism

$$\Phi : \mathrm{Gal}(\Omega/K) \to \hat{\mathbb{Z}}^{(p)}(1)$$

here π is the generator of the maximal ideal of $R(s)$. Now we prove that $\mathrm{Gal}(\Omega/K)$ is isomorphic to $\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega})$. Let us have a finite, étale, Galois cover Y of $D(s) \smallsetminus \{s\}$, and we have the morphism

$$\mathrm{Spec}(K) \to D(s) \smallsetminus \{s\}$$

consider the following Cartesian square.

$$\begin{array}{ccc} Y' = Y \times_{D(s) \smallsetminus \{s\}} \mathrm{Spec}(K) & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ \mathrm{Spec}(K) & \longrightarrow & D(s) \smallsetminus \{s\} \end{array}$$

Then since the base change of an étale morphism is étale the morphism

$$Y \times_{D(s) \smallsetminus \{s\}} \mathrm{Spec}(K) \to \mathrm{Spec}(K)$$

is étale and since the base change is happening over $\mathrm{Spec}(K)$, and since Y is finite over $D(s) \smallsetminus \{s\}$ we get that, $D(s) \smallsetminus \{s\}$ is covered by $V_i = \mathrm{Spec}(B_i)$'s such that $g^{-1}(V_i) = \mathrm{Spec}(A_i)$ and A_i is a finitely generated B_i module. This gives us that $A_i \otimes_{B_i} K$ is a finitely generated K-vector space, in fact a finitely generated K algebra. Therefore we get an integral scheme $\mathrm{Spec}(A_i \otimes_{B_i} K)$ over $\mathrm{Spec}(K)$, this gives us that the $A_i \otimes_{B_i} K$ is an integral domain and hence a finite field extension of K. Therefore the function field $K(Y')$ is a finite field extension of K and any automorphism σ of Y gives us an automorphism of Y', which gives us an automorphism σ of $K(Y')$ over K. Therefore we have a morphism from

$$Aut(Y/D(s) \smallsetminus \{s\}) \to Aut(K(Y')/K)$$

therefore passing to the inverse limit we have that

$$\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \to \mathrm{Gal}(\Omega/K)\ .$$

On the other hand suppose that we have a finite extension L of K such that the normalization of $D(s) \smallsetminus \{s\}$ in L is unramified, call it Y. Then Y is flat and unramified over $D(s) \smallsetminus \{s\}$, therefore it is étale. It is easy to see that it is finite. Any automorphism σ of L gives rise to an automorphism of Y over $D(s) \smallsetminus \{s\}$. This gives us the inverse map

$$\mathrm{Gal}(\Omega/K) \to \pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \ .$$

Now we have the following commutative diagram

$$\begin{array}{ccc} \pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) & \longrightarrow & \pi_1(D \smallsetminus A, \omega) \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}^{(p)}(1) & \longrightarrow & \pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega) \end{array}$$

by A.I.13 in [12]. Let $\mathfrak{a}_s$ be the kernel of the morphism

$$\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \to \hat{\mathbb{Z}}^{(p)}(1)$$

and the groups $\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega})$ generate the group $\pi_1(D \smallsetminus A, \omega)$, therefore since the group $\pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega)$ is a quotient of $\pi_1(D \smallsetminus A, \omega)$, and the above diagram is commutative we get that the kernel of the map

$$\pi_1(D \smallsetminus A, \omega) \to \pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega)$$

is generated by the images of the kernels $\mathfrak{a}_s$ of the map

$$\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \to \pi_1(D \smallsetminus A, \omega)$$

and the tame fundamental group is the quotient of $\pi_1(D \smallsetminus A, \omega)$ by the smallest normal subgroup that contains all images of the kernels $\mathfrak{a}_s$. A continuous representation of $\pi_1(D \smallsetminus A, \omega)$ on a finite dimensional vector space V over $\mathbb{Q}_l$ respectively the $\pi_1(D \smallsetminus A, \omega)$ module V is said to be tamely ramified if the representation factors through $\pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega)$. That is equivalent to say that the kernels $\mathfrak{a}_s$ acts trivially on the vector space V for all $s \in A$ and for all $\widetilde{\omega}$. A locally constructible sheaf $\mathscr{F}$ on $D \smallsetminus A$ is called tamely ramified when the stalk $\mathscr{F}_\omega$ is a tame $\pi_1(D \smallsetminus A, \omega)$ module. Every point $s \in A$ gives rise to a local Lefschetz pencil of the following type.

$$\begin{array}{ccc} \widetilde{X} \times_D D(s) & \longrightarrow & \widetilde{X} \\ {\scriptstyle f_{D(s)}} \downarrow & & \downarrow {\scriptstyle f} \\ D(s) & \longrightarrow & D = \mathbb{P}^1 \end{array}$$

The generic fiber respectively the special fiber over s is isomorphic to the generic fiber over ω and $f^{-1}(s)$. By the proper base change theorem $R^i f_*(\mathbb{Q}_{l\widetilde{X}})_\omega$ is canonically isomorphic to $H^i_{\acute{e}t}(\widetilde{X_\omega}, \mathbb{Q}_l)$ and likewise $R^i f_*(\mathbb{Q}_{l\widetilde{X}})_s$ is isomorphic to $H^i_{\acute{e}t}(\widetilde{X_s}, \mathbb{Q}_l)$. Now we formulate the Picard-Lefschetz formulas.

Theorem 3.5.2. *Let V be a $\pi_1(D \setminus A, \omega)$ module $R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega = H^n_{ét}(\widetilde{X_\omega}, \mathbb{Q}_l)$ then we have the following.*

(I) The sheaves $R^i f_*(\mathbb{Q}_{l\widetilde{X}})$ are locally constant for $i \neq n, n+1$ hence constant on D, since the fundamental group of D vanishes.

(II) For each $s \in A$ there is a vanishing cycle δ_s in $V(m)$ that depends upto conjugation only on s and not on the choice of $\widetilde{\omega}$, a cohomology class δ_s^* in $R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})_s(n-m)$ and an exact sequence between the specialization mappings with respect to $\widetilde{\omega}$.

$$0 \to R^n f_*(\mathbb{Q}_{l\widetilde{X}})_s \to R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega \to \mathbb{Q}_l(m-n) \to$$

$$R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})_s \to R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})_\omega \to$$

where the map

$$R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega \to \mathbb{Q}_l(m-n)$$

is

$$a \mapsto \langle a, \delta_s \rangle$$

and the map

$$\mathbb{Q}_l(m-n) \to R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})_s$$

is

$$\lambda \mapsto \lambda \delta_s^* \ .$$

(III) The sheaves $R^n f_*(\mathbb{Q}_{l\widetilde{X}})$ and $R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})$ are locally constant on $D \setminus A$. $\pi_1(D \setminus A, \omega)$ acts trivially on $R^{n+1} f_*(\mathbb{Q}_{l\widetilde{X}})_\omega$ and tamely on $R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega$. For $x \in R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega$ and $u \in \hat{\mathbb{Z}}^{(p)}(1)$ we have that

$$\gamma_s(u)(x) = x - (-1)^m \bar{u} \langle x, \delta_s \rangle \delta_s$$

here $\bar{u}$ is the natural image of u in $\mathbb{Z}_l(1)$.

First we discuss what is $\bar{u}$. Let us consider the natural homomorphisms

$$\varprojlim \mu_n \to \mu_n$$

and then consider the map

$$\mu_n \to (\mu_n)_X$$

given as follows. First we have

$$(\mu_n)_X(X') = \{s | s^n = 1\} \ ,$$

let a be an n-th root of unity, then define the regular map

$$s_a : X' \to \mathbb{A}^1(k)$$

by

$$s_a(u) = a \ .$$

Then it is clear that we have

$$s_a^n = 1 \ .$$

That gives us a morphism from μ_n to $(\mu_n)_X$ as a morphism of presheaves. Now the map

$$a \mapsto a^l$$

from

$$\mu_{l^{n+1}} \to \mu_{l^n}$$

is such that

$$\begin{array}{ccc} \varprojlim \mu_{l^n} & \longrightarrow & \mu_{l^{n+1}} \\ & \searrow & \downarrow \\ & & \mu_{l^n} \end{array}$$

is commutative and we get the diagram

$$\begin{array}{ccc} \varprojlim \mu_{l^n} & \longrightarrow & (\mu_{l^{n+1}})_X \\ & \searrow & \downarrow \\ & & (\mu_{l^n})_X \end{array}$$

is commutative therefore by the universality of the inverse limit we get a unique map from

$$\varprojlim \mu_{l^n} \to \mathbb{Z}_l(1) \ .$$

We denote this by $\bar{u}$.

Proof. Since f is smooth and proper over $D \smallsetminus A$ all the sheaves $R^i f_*(\mathbb{Q}_{l\widetilde{X}})$ are locally constant on $D \smallsetminus A$ by the theorem 8.9 in chapter I in [12]. Now consider a local Lefschetz pencil

$$\widetilde{X} \times_D D(s) \stackrel{f_{D(s)}}{\to} D(s)$$

then since $D(s)$ is strictly Henselian valuation ring and the point s is such that $f_{D(s)}^{-1}(s)$ is singular and has one ordinary double point. Then we can use the theorem 4.3 in chapter III in [12] which says the following. Let $f : X \to S$ be a flat, proper morphism where $S = \mathrm{Spec}(R)$, R is a strictly Henselian valuation ring and f is of odd fiber dimension. Let f be singular at just one point s, then

$$H^i_{\acute{e}t}(X_s, \mathbb{Q}_l) \to H^i_{\acute{e}t}(X_\omega, \mathbb{Q}_l)$$

is an isomorphism for $i \neq n, n+1$, where ω is the geometric generic point of S. Then by applying this theorem to the local Lefschetz pencil

$$\widetilde{X} \times_D D(s) \overset{f_{D(s)}}{\to} D(s)$$

we get that

$$H^i_{\acute{e}t}(X_s, \mathbb{Q}_l) \cong H^i_{\acute{e}t}(X_\omega, \mathbb{Q}_l)$$

for all $i \neq n, n+1$, therefore the sheaf $R^i f_*(\mathbb{Q}_{l\widetilde{X}})$ is locally constant on D by lemma 8.12 in chapter I in [12] which says the a sheaf $\mathscr{G}$ is locally constant on a scheme S if and only if all the specialization homomorphisms are isomorphisms and all stalks of $\mathscr{G}$ are finite. Therefore we get that $R^i f_*(\mathbb{Q}_{l\widetilde{X}})$ is locally constant on D. II) follows from theorem 4.3 in chapter 3 in [12]. Now by the same theorem 4.3 in chapter 3 in [12] we have that $\mathrm{Gal}(\Omega/K)$ acts trivially on $H^{n+1}(\widetilde{X_\omega}, \mathbb{Q}_l)$. Since $\mathrm{Gal}(\Omega/K)$ is isomorphic to $\pi_1(D(s) \smallsetminus \{s\})$ we have that $\pi_1(D(s) \smallsetminus \{s\})$ is acting trivially on $H^{n+1}(\widetilde{X_\omega}, \mathbb{Q}_l)$. These groups and their conjugates generate the group $\pi_1(D \smallsetminus A, \omega)$. Therefore $\pi_1(D \smallsetminus A, \omega)$ is acting trivially on $H^{n+1}(\widetilde{X_\omega}, \mathbb{Q}_l)$. The action of $\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega})$, that is the action of $\mathrm{Gal}(\Omega/K)$ on $H^n(\widetilde{X_\omega}, \mathbb{Q}_l)$ factors through the homomorphism

$$\Phi : \pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \to \hat{\mathbb{Z}}^{(p)}(1)$$

and again we use the commutative diagram

$$\begin{array}{ccc} \pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) & \longrightarrow & \pi_1(D \smallsetminus A, \omega) \\ \downarrow & & \downarrow \\ \hat{\mathbb{Z}}^{(p)}(1) & \longrightarrow & \pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega) \end{array}$$

and observe the fact that the kernel $\mathfrak{a}_s$ of the map

$$\pi_1(D(s) \smallsetminus \{s\}, \widetilde{\omega}) \to \hat{\mathbb{Z}}^{(p)}(1)$$

acts trivially on $H^n(\widetilde{X_\omega}, \mathbb{Q}_l)$, therefore the action of $\pi_1(D \smallsetminus A, \omega)$ factors through the action of $\pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega)$. So we are done. □ Set,

$$E = \sum_{s \in A, \sigma \in \pi_1^{\mathrm{tame}}(D \smallsetminus A, \omega)} \mathbb{Q}(-m)\sigma(\delta_s)$$

this is called the space of vanishing cycles in V.

Proposition 3.5.3. *All the vanishing cycles δ_s are conjugate up to sign. In particular E vanishes if and only if all vanishing cycles δ_s are zero.*

Proof. We embed the Lefschetz pencil $\widetilde{X} \to D$ in the family of hyperplane sections of X.

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & Z \subset X \times \mathbb{P}^{N^\vee} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ D & \longrightarrow & \mathbb{P}^{N^\vee} \end{array}$$

Here

$$Z = \{(x, H) : x \in H\}$$

and the map g is smooth outside the dual variety F of X, that is the variety consisting of all hyperplanes that touches X in atleast one point. Denote $F \cap D = A$, we can assume that the codimension of F is one otherwise $F \cap D = A$ is empty. F meets D transversally and only in smooth points. By definition of a Lefschetz embedding there exists a closed subset F' of codimension greater or equal than 2, such that for all points $x \in F \setminus F'$, we have the fiber g has exactly one ordinary double point. Now consider a point a in $\mathbb{P}^{N^\vee} \setminus (F \setminus F')$ and consider the local ring $\mathscr{O}_{\mathbb{P}^{N^\vee}, a}$ and its strict Henselization $\widetilde{\mathscr{O}^{\vee}_{\mathbb{P}^N}, a}$ and the spectrum of it, call it $D(a)$. Then consider the Cartesian square.

$$\begin{array}{ccc} \widetilde{X} \times_{\mathbb{P}^{N^\vee}} D(a) & \longrightarrow & \widetilde{X} \\ {\scriptstyle g_{D(a)}}\downarrow & & \downarrow{\scriptstyle g} \\ D(a) & \longrightarrow & \mathbb{P}^{N^\vee} \end{array}$$

Let $\omega, \widetilde{\omega}$ be as before. Now by the theorem 4.3 in chapter 3 of [12] we get that the action of

$$\pi_1(D(a) \setminus \{a\}, \widetilde{\omega})$$

on $R^n g_*(\mathbb{Q}_l)$ factors through the homomorphism

$$\pi_1(D(a) \setminus \{a\}) \to \hat{\mathbb{Z}}^{(p)}(1)$$

therefore we have that $\pi_1(\mathbb{P}^{N^\vee} \setminus (F \setminus F'))$ acts tamely on $R^n g_*(\mathbb{Q}_l)_\omega$ and $\pi_1^{\text{tame}}(\mathbb{P}^{N^\vee} \setminus F)$ is a subgroup of $\pi_1^{\text{tame}}(\mathbb{P}^{N^\vee} \setminus (F \setminus F'))$, and therefore $\pi_1(\mathbb{P}^{N^\vee} \setminus F)$ acts tamely on $R^n g_*(\mathbb{Q}_l)_\omega$. Following A.I theorem 16 in [12] we get a surjective map

$$q : \pi_1^{\text{tame}}(D \setminus A, \omega) \to \pi_1^{\text{tame}}(\mathbb{P}^{N^\vee} \setminus F, \omega) .$$

Now consider the Cartesian square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;i\;} & Z \subset X \times \mathbb{P}^{N\vee} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow{\;j\;} & \mathbb{P}^{N\vee} \end{array}$$

then by the proper base change theorem it follows that

$$R^n f_*(\mathbb{Q}_{l\widetilde{X}})_\omega = R^n g_*(\mathbb{Q}_{l\widetilde{X}})_\omega = V$$

therefore the action of $\pi_1^{\text{tame}}(D \smallsetminus A, \omega)$ factors through q. On the other hand all the homomorphisms

$$q \circ \gamma_s : \hat{\mathbb{Z}}^{(p)}(1) \to \pi_1^{\text{tame}}(\mathbb{P}^{N\vee} \smallsetminus F)$$

are conjugate in $\pi_1^{\text{tame}}(\mathbb{P}^{N\vee} \smallsetminus F)$ due to theorem A.I.16 in [12]. Thus for s, s' in A there is an element $\sigma \in \pi_1^{\text{tame}}(D \smallsetminus A, \omega)$ with

$$\gamma_s'(u)(x) = \sigma\gamma_s(u)\sigma^{-1}(x)$$

for all $u \in \hat{\mathbb{Z}}^{(p)}(1)$ and $x \in V$. Therefore from the Picard-Lefschetz formula it follows that

$$\gamma_s'(u)(x) = x - (-1)^m \bar{u}\langle\sigma^{-1}(x), \delta_s\rangle\delta_s$$

for all $u \in \hat{\mathbb{Z}}^{(p)}(1)$ and $x \in V$. Thus we have that

$$\sigma\delta_s = \pm\delta_{s'} \ .$$

□

Corollary 3.5.4. *The induced representation of* $\pi_1^{\text{tame}}(D \smallsetminus A, \omega)$ *on* $E/E \cap E^\perp$ *is absolutely irreducible.*

Proof. Let L be an extension field of $\mathbb{Q}_l$ and M a $\pi_1^{\text{tame}}(D \smallsetminus A, \omega)$ stable vector subspace of $E \otimes L$ that is not contained in $(E \cap E^\perp) \otimes L$. Then there is an element x in M, and there is some δ_s such that

$$\langle x, \delta_s\rangle \neq 0 \ .$$

Then by the Picard-Lefschetz formula we have that

$$\gamma_s(u)x - x = (-1)^{m+1}\langle x, \delta_s\rangle\delta_s$$

which gives us that δ_s belongs to M, since the vanishing cycles are conjugate to each other upto sign, this gives us that δ_s belongs to M for all s in A. That proves that $M = E \otimes L$. □

Chapter 4

Algebraic cycles on nonsingular cubics in $\mathbb{P}^5$

4.1 Chow schemes and countability results

Fix a field k of characteristic zero, and take the category $\mathfrak{N}$ of all Noetherian schemes over k. Then we can consider the presheaf of abelian monoids $\mathscr{C}_r(X)$ on $\mathfrak{N}$, such that for any scheme S in $\mathfrak{N}$ the monoid

$$\mathscr{C}_r(X)(S) = \mathscr{C}_r(X \times S/S)$$

is freely generated by relative cycles on $X \times S$ of (relative) dimension r over S, see [35].

If X is equidimensional, then we will also write $\mathscr{C}^p(X)(S)$ or $\mathscr{C}^p(X \times S/S)$ for the same monoids of relative cycles of relative codimension p, where $p = \dim(X) - r$.

If X is projective over k, we fix a closed embedding of X into $\mathbb{P}^m$ and consider the subpresheaf $\mathscr{C}^p_d(X)$ of relative cycles of degree d in $\mathscr{C}^p(X)$.

Since the characteristic of k is zero, the presheaf $\mathscr{C}^p_d(X)$ on $\mathfrak{N}$ is representable by the Chow scheme $C^p_d(X)$ projective over k, see [22] or [35]. In other words, for each S one has the bijection

$$\theta_X(S) : \mathscr{C}^p(X \times S/S) \stackrel{\sim}{\to} \mathrm{Hom}(S, C^p(X)) \ ,$$

functorial in S. The bijections θ are functorial in X due to Corollary 3.6.3 in [35]. In case X is equi-dimensional, we may also write $C_{d,r}(X)$ instead of $C^p_d(X)$, where $r = \dim(X) - p$. Let also $C^p(X)$ or $C_r(X)$ be the coproducts of the corresponding Chow schemes for all $d \geq 0$.

It is trivial but worth noticing that if k' is another field and

$$\alpha : k \stackrel{\sim}{\to} k'$$

is an isomorphism of fields, the scheme $C_r(X')$ is the pull-back of the scheme $C_r(X)$ with respect to the morphism $\mathrm{Spec}(\alpha)$, where X' is the pull-back of X, and the corresponding morphism from $C^p(X')$ to $C^p(X)$ is an isomorphism of schemes. The bijections $\theta_{X'}$ and θ_X commute by means of the obvious isomorphisms on monoids and Hom-sets induced by the isomorphism α.

In what follows the word "monoid" will always mean the commutative (i.e. abelian) monoid written additively.

For a monoid M its group completion M^+ is the minimal group arising from M, i.e. the value of the left adjoint to the forgetful functor from groups to monoids. It can be constructed in several fairly different ways. We prefer the following one. Consider the quotient of $M \oplus M$ by the image of the diagonal embedding. Let τ be the corresponding quotient homomorphism, and let ν be the composition of the embedding of M as one of the two direct summands and the homomorphism τ. Then

$$\nu : M \to M^+$$

possesses the obvious universal property, and for any (a, b) in $M \oplus M$ the value $\tau(a, b)$ is the difference $\nu(a) - \nu(b)$.

Notice that if M is a cancellation monoid then ν is injective, and we can identify M with its image in M^+. Modulo this identification,

$$\tau(a, b) = a - b\ .$$

In particular, we can consider the presheaf $\mathscr{Z}^p(X)$ of abelian groups on $\mathfrak{N}$, such that for each S the group of sections $\mathscr{Z}^p(X \times S/S)$ is the completion

$$\mathscr{C}^p(X \times S/S)^+$$

of the monoid $\mathscr{C}^p(X \times S/S)$.

The Chow scheme

$$C^p(X) = \coprod_{d \geq 0} C^p_d(X)$$

is naturally a commutative monoid, which can be completed getting the abelian group

$$Z^p(X) = C^p(X)^+$$

with the attached homomorphisms τ and ν.

We will also be using the schemes of morphisms from one noetherian scheme to another. More precisely, if S and Y are two Noetherian schemes over the ground field k, then we can consider the functor

$$\mathscr{H}om(S, Y)$$

on $\mathfrak{N}$ sending a scheme T to the set $\mathscr{H}om(S, Y)(T)$ of morphisms from $S \times T$ to $Y \times T$ over T. This is a subfunctor of the Hilbert functor

$$\mathscr{H}ilb(S \times Y)$$

via the graphs of the morphisms between schemes.

It is well-known that if S and Y are projective over k, the Hilbert functor is representable by the projective Hilbert scheme $\mathrm{Hilb}(S \times Y)$ and $\mathscr{H}om(S,Y)$ is representable by an open subscheme $\mathrm{Hom}(S,Y)$ in $\mathrm{Hilb}(S \times Y)$, see [11]

Let now d be a positive integer. Then one can choose a Hilbert polynomial Φ_d, such that the set-theoretic intersection

$$\mathscr{H}om^d(S,Y) = \mathscr{H}om(S,Y) \cap \mathscr{H}ilb_{\Phi_d}(S \times Y)$$

inside $\mathscr{H}ilb(S \times Y)$ would be representable by an open subscheme $\mathrm{Hom}^d(S,Y)$ in the projective scheme $\mathrm{Hilb}_{\Phi_d}(S \times Y)$. We recall the construction of $\mathscr{H}ilb_{\Phi_d}(S \times Y)$ in brief. Let $\mathscr{O}(1)$ be a relatively ample line bundle on $S \times Y$ and let Φ_d be a polynomial, then the functor $\mathscr{H}ilb_{\Phi_d}(S \times Y)$ from schemes over S to sets is defined as follows. For scheme Z over S we have $\mathscr{H}ilb_{\Phi_d}(S \times Y)(Z)$ is the set of closed subschemes V of $Z \times_S S \times Y$, which are proper and flat over Z and have Hilbert polynomial Φ_d. The Hilbert polynomial of the closed subscheme V of $Z \times_S S \times Y$ is the Hilbert polynomial of $\mathscr{O}_V$.

In particular, if $S = \mathbb{P}^1$ and Y is embedded into some $\mathbb{P}^n$ then we obtain a quasiprojective scheme

$$\mathrm{Hom}^d(\mathbb{P}^1, Y)$$

over the ground field k parametrizing rational curves of degree d in Y.

By the universal property of fibred products over k one has the natural bijection between $\mathscr{H}om(S,Y)(T)$ and $\mathrm{Hom}(T \times S, Y)$. This gives the adjunction

$$\mathrm{Hom}(T \times S, Y) \simeq \mathrm{Hom}(T, \mathrm{Hom}(S,Y)) \ .$$

In ones turn, this adjunction induces the regular evaluation morphism

$$e_{S,Y} : \mathrm{Hom}(S,Y) \times S \to Y \ .$$

The latter also induces the regular evaluation morphism of quasi-projective schemes

$$e_{S,Y} : \mathrm{Hom}^d(S,Y) \times S \to Y \ ,$$

for each positive integer d. In particular, if P is a closed point of $\mathbb{P}^1$ one has the regular evaluation morphism

$$e_P : \mathrm{Hom}^d(\mathbb{P}^1, Y) \to Y$$

sending f to $f(P)$. This all can be found in [22].

Being equipped with the above tools, we can now study rational equivalence of algebraic cycles in terms of rational curves on appropriate Chow schemes.

So, take a nonsingular projective X and embed it into the projective space $\mathbb{P}^n$ over k. Let A and A' be two algebraic cycles of codimension p on X. As we have already mentioned in Introduction, the cycle A is rationally equivalent to the cycle A' if and only if there exists an effective, i.e. positive, algebraic cycle Z on $X \times \mathbb{P}^1$ and an effective algebraic cycle B on X, such that

$$Z(0) = A + B \quad \text{and} \quad Z(\infty) = A' + B \ .$$

Both Z and B are of codimension p in $X \times \mathbb{P}^1$ and X respectively.

Now, assume that A is rationally equivalent to A', and let

$$f_Z = \theta(Z) \quad \text{and} \quad f_{B\times\mathbb{P}^1} = \theta(B \times \mathbb{P}^1)$$

be two regular morphisms from $\mathbb{P}^1$ to $\mathscr{C}^p(X)$. Here $\theta = \theta_X(\mathbb{P}^1)$ is the above bijection from $\mathscr{C}^p(X \times \mathbb{P}^1/\mathbb{P}^1)$ to $\mathrm{Hom}(\mathbb{P}^1, C^p(X))$. Let also

$$f = f_Z \oplus f_{B\times\mathbb{P}^1}$$

be the morphism from $\mathbb{P}^1$ to $C^p(X) \oplus C^p(X)$ generated by f_Z and $f_{B\times\mathbb{P}^1}$.

Notice that as $C^p(X)$ is a cancellation monoid, for any two elements $a, b \in C^p(X)$ the value $\tau(a, b)$ in $C^p(X)^+$ is $a - b$. Here we identify $C^p(X)$ with its image in $C^p(X)^+$ under the injective homomorphism ν.

Then

$$\tau f(0) = \tau(f_Z(0), f_{B\times\mathbb{P}^1}) = f_Z(0) - f_{B\times\mathbb{P}^1}(0) = Z(0) - B = A$$

and

$$\tau f(\infty) = \tau(f_Z(\infty), f_{B\times\mathbb{P}^1}) = f_Z(0) - f_{B\times\mathbb{P}^1}(\infty) = Z(\infty) - B = A' \ .$$

Vice versa, if there is a regular morphism

$$f = f_1 \oplus f_2 : \mathbb{P}^1 \to C^p(X) \oplus C^p(X) \ ,$$

such that $\tau f(0) = A$ and $\tau f(\infty) = A'$, by setting Z_1 and Z_2 to be two algebraic cycles in $\mathscr{C}^p(X \times \mathbb{P}^1/\mathbb{P}^1)$, such that $\theta(Z_i) = f_i$ for $i = 1, 2$, and

$$Z = Z_1 - Z_2 \ ,$$

we obtain that $Z(0) = A$ and $Z(\infty) = A'$. It means that A is rationally equivalent to A'. Recall that we fix an embedding X inside some $\mathbb{P}^m$.

Now, for any non-negative integers $d_1, \ldots, d_s$ let

$$C^p_{d_1,\ldots,d_s}(X)$$

be the product

$$C^p_{d_1}(X) \times \cdots \times C^p_{d_s}(X)$$

over k. Following the work of Roitman, [28], for any degree $d \geq 0$ we define

$$W_d$$

to be the set of ordered pairs

$$(A, B) \in C^p_{d,d}(X) \ ,$$

such that the cycle A is rationally equivalent to the cycle B on X. For any non-negative u and positive v let also

$$W_d^{u,v}$$

be the subset of closed points (A, B) in $C^p_{d,d}(X)$, such that there exists

$$f \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X))$$

with $\tau f(0) = A$ and $\tau f(\infty) = B$. Then

$$W_d^{u,v} \subset W_d$$

and

$$W_d = \cup_{u,v} W_d^{u,v} \ .$$

Let also $\bar{W}_d^{u,v}$ be the Zariski closure of the set $W_d^{u,v}$ in the projective scheme $C^p_{d,d}(X)$.

The following proposition is a straightforward generalization of the result in [28] migrated to Roitman's paper from the famous paper [24].

Proposition 4.1.1. *For any d, u and v the set $W_d^{u,v}$ is a quasi-projective subscheme in $C^p_{d,d}(X)$ whose Zariski closure $\bar{W}_d^{u,v}$ is contained in W_d.*

Proof. Let

$$e : \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \to C^p_{d+u,u,d+u,u}(X)$$

be the evaluation morphism sending $f : \mathbb{P}^1 \to C^p_{d+u,u}(X)$ to the ordered pair $(f(0), f(\infty))$, and let

$$s : C^p_{d,u,d,u}(X) \to C^p_{d+u,u,d+u,u}(X)$$

be the regular morphism sending (A, C, B, D) to $(A + C, C, B + D, D)$. The two morphisms e and s allow to consider the fibred product

$$V = \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times_{C^p_{d+u,u,d+u,u}(X)} C^p_{d,u,d,u}(X) \ .$$

This V is a closed subvariety in the product

$$\mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times C^p_{d,u,d,u}(X)$$

over $\mathrm{Spec}(k)$ consisting of quintuples (f, A, C, B, D) such that

$$e(f) = s(A, C, B, D) \ ,$$

i.e.

$$(f(0), f(\infty)) = (A + C, C, B + D, D) .$$

The latter equality gives

$$\mathrm{pr}_{2,3}(V) \subset W_d^{u,v} .$$

where $\mathrm{pr}_{2,3}$ is the projection of the product $\mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \times C^p_{d,u,d,u}(X)$ onto $C^p_{d,d}(X)$.

Vice versa, if (A, B) is a closed point of $W_d^{u,v}$, there exists a regular morphism

$$f \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X))$$

with $\tau f(0) = A$ and $\tau f(\infty) = B$.

Let $f(0) = (C', C)$ and $f(\infty) = (D', D)$. Then

$$\tau f(0) = C' - C = A \qquad \text{and} \qquad \tau f(\infty) = D' - D = B$$

in the completed monoid $Z^p(X) = C^p(X)^+$. In other words, there exist effective codimension p algebraic cycles M and N on X, such that

$$C' + M = C + A + M \qquad \text{and} \qquad D' + N = D + B + N$$

in $C^p(X)$.

Since $C^p(X)$ is a free monoid, it possesses the cancellation property. Therefore, $C' + M = C + A + M$ implies $C' = C + A$ and $D' + N = D + B + N$ implies $D' = D + B$. This yields $e(f) = s(A, C, B, D)$, whence

$$(f, A, C, B, D) \in V .$$

It means that (A, B) is in $\mathrm{pr}_{2,4}(V)$.

Thus,

$$\mathrm{pr}_{2,3}(V) = W_d^{u,v} .$$

Being the image of a quasi-projective variety under the projection $\mathrm{pr}_{2,4}$ the set $W_d^{u,v}$ is itself a quasi-projective variety.

Let

$$\tilde{s} : C^p_{d,d,u,u}(X) \to C^p_{d+u,d+u,u,u}(X)$$

be the morphism obtained by composing and precomposing s with the transposition of the second and third factors in the domain and codomain of the above morphism s. Then

$$W_d = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(W_{d+u} \times W_u)) .$$

Let (A, B, C, D) be a closed point in $C^p_{d,d,u,u}(X)$, such that

$$\tilde{s}(A, B, C, D) = (A + C, B + D, C, D)$$

is in $W^{0,v}_{d+u} \times W^{0,v}_{u}$. The latter condition means that there exist two regular morphisms

$$g \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u}(X))$$

and

$$h \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{u}(X))$$

with

$$g(0) = A + C\ , \quad g(\infty) = B + D\ , \quad h(0) = C \quad \text{and} \quad h(\infty) = D\ .$$

Let $f = g \times h$. Then

$$f \in \mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X))\ ,$$

$$f(0) = (A + C, C)$$

and

$$f(\infty) = (B + D, D)\ .$$

Hence,

$$\tau f(0) = A \qquad \text{and} \qquad \tau f(\infty) = B\ .$$

It means that

$$(A, B) \in W^{u,v}_{d}\ .$$

Thus, we have shown that

$$\mathrm{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_{u})) \subset W^{u,v}_{d}\ .$$

Vice versa, take a point (A, B) in $W^{u,v}_{d}$ and let f be in $\mathrm{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X))$, such that $\tau f(0) = A$ and $\tau f(\infty) = B$. Composing f with the projections of $C^p_{d+u,u}(X)$ onto $C^p_{d+u}(X)$ and $C^p_{u}(X)$ one can show that $W^{u,v}_{d}$ is a subset in $\mathrm{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_{u}))$. Thus,

$$W^{u,v}_{d} = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_{u}))\ .$$

Since $\tilde{s}$ is continuous and $\mathrm{pr}_{1,2}$ is proper,

$$\bar{W}^{u,v}_{d} = \mathrm{pr}_{1,2}(\tilde{s}^{-1}(\bar{W}^{0,v}_{d+u} \times \bar{W}^{0,v}_{u}))\ .$$

This gives that to prove the second assertion of the proposition it is enough to show that $\bar{W}^{0,v}_{d}$ is contained in W_d.

Let (A, B) be a closed point of $\bar{W}^{0,v}_{d}$. If (A, B) is in $W^{0,v}_{d}$, then it is also in W_d. Suppose

$$(A, B) \in \bar{W}^{0,v}_{d} \smallsetminus W^{0,v}_{d}\ .$$

Let W be an irreducible component of the quasi-projective variety $W^{0,v}_{d}$ whose Zariski closure $\bar{W}$ contains the point (A, B). Let U be an affine neighbourhood of (A, B) in $\bar{W}$. Since (A, B) is in the closure of W, the set $U \cap W$ is non-empty.

Let us show that we can always take an irreducible curve C passing through (A, B) in U. Indeed, write U as $\mathrm{Spec}(A)$. It is enough to show that there exists a prime ideal in $\mathrm{Spec}(A)$ of height $n-1$, where n is the dimension of $\mathrm{Spec}(A)$, where A is Noetherian. Since A is of dimension n there exists a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

such that this chain can not be extended further. Now consider the subchain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n-1} .$$

This is a chain of prime ideals and $\mathfrak{p}_{n-1}$ is a prime ideal of height $n-1$, so we get an irreducible curve.

Let $\bar{C}$ be the Zariski closure of C in $\bar{W}$. Two evaluation regular morphisms e_0 and e_∞ from $\mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X))$ to $C^p_d(X)$ give the regular morphism

$$e_{0,\infty} : \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \to C^p_{d,d}(X) .$$

Then $W^{0,v}_d$ is exactly the image of the regular morphism $e_{0,\infty}$, and we can choose a quasi-projective curve T in $\mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X))$, such that the closure of the image $e_{0,\infty}(T)$ is $\bar{C}$.

For that consider the curve C in W so it is contained in $W^{0,v}_d$. We know that the image of $e_{0,\infty}$ is $W^{0,v}_d$. Consider the inverse image of $\bar{C}$ under the morphism $e_{0,\infty}$. Since $\bar{C}$ is a curve, the dimension of $e^{-1}_{0,\infty}(C)$ is greater than or equal than 1. So it contains a curve. Consider two points on $\bar{C}$, consider their inverse images under $e_{0,\infty}$. Since $\mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X))$ is a quasi projective variety, $e^{-1}_{0,\infty}(\bar{C})$ is also projective, we can embed it into some $\mathbb{P}^m$ and consider a smooth hyperplane section through the two points fixed above. Continuing this procedure we get a curve containing these two points and contained in $e^{-1}_{0,\infty}(C)$. Therefore we get a curve T mapping onto $\bar{C}$. So the closure of the image of T is $\bar{C}$.

Now, as we have mentioned above, $\mathrm{Hom}^v(\mathbb{P}^1, \mathscr{C}^p_d(X))$ is a quasi-projective variety. This is why we can embed it into some projective space $\mathbb{P}^m$. Let $\bar{T}$ be the closure of T in $\mathbb{P}^m$, let $\tilde{T}$ be the normalization of $\bar{T}$ and let $\tilde{T}_0$ be the pre-image of T in $\tilde{T}$. Consider the composition

$$f_0 : \tilde{T}_0 \times \mathbb{P}^1 \to T \times \mathbb{P}^1 \subset \mathrm{Hom}^v(\mathbb{P}^1, C^p_d(X)) \times \mathbb{P}^1 \xrightarrow{e} C^p_d(X) ,$$

where e is the evaluation morphism $e_{\mathbb{P}^1, C^p_d(X)}$. The regular morphism f_0 defines a rational map

$$f : \tilde{T} \times \mathbb{P}^1 \dashrightarrow C^p_d(X)$$

Since $\tilde{T}$ is a non-singular projective curve, the product $\tilde{T} \times \mathbb{P}^1$ is a non-singular projective surface over the ground field. Under this condition there exists a finite chain of σ-processes $(\tilde{T} \times \mathbb{P}^1)' \to \tilde{T} \times \mathbb{P}^1$ resolving indeterminacy of f and giving a regular morphism

$$f' : (\tilde{T} \times \mathbb{P}^1)' \to C^p_d(X) .$$

The regular morphism $\tilde{T}_0 \to T \to \bar{C}$ extends to the regular morphism $\tilde{T} \to \bar{C}$. Let P be a point in the fibre of this morphism at (A, B). For any closed point Q on $\mathbb{P}^1$ the restriction $f|_{\tilde{T}\times\{Q\}}$ of the rational map f onto $\tilde{T} \times \{Q\} \simeq \tilde{T}$ is regular on the whole curve $\tilde{T}$, because $\tilde{T}$ is non-singular. Then

$$(f|_{\tilde{T}\times\{0\}})(P) = A \qquad \text{and} \qquad (f|_{\tilde{T}\times\{\infty\}})(P) = B\ .$$

It means that the points A and B are connected by a finite collection of curves which are the images of rational curves on $(\tilde{T} \times \mathbb{P}^1)'$ under the regular morphism f'. In turn, it follows that A is rationally equivalent to B, whence

$$(A, B) \in W_d\ .$$

□

In what follows, for any equi-dimensional algebraic scheme V let $CH^p(V)$ be the Chow group, with coefficients in $\mathbb{Z}$, of codimension p algebraic cycles modulo rational equivalence on V. Let then

$$\theta_d^p : C_{d,d}^p(X) \to CH^p(X)$$

be the map sending (P, Q) to the class of the difference $Z_P - Z_Q$, where Z_P and Z_Q are degree d and codimension p cycles on X corresponding to the points P and Q respectively.

Corollary 4.1.2. $(\theta_d^p)^{-1}(0)$ *is a countable union of irreducible Zariski closed subsets in the Chow scheme* $C_{d,d}^p(X)$.

Proof. Proposition 4.1.1 gives that W_d is the countable union of Zariski closed sets $\bar{W}_d^{u,v}$ over u and v. This completes the proof. □

4.2 Proper push-forward on Chow groups

Let k be an algebraically closed uncountable field of characteristic zero. All schemes in this section will be either over k or over the residue fields of points of schemes over k. For an algebraic variety Y over k, let $CH^p(Y)$ be the codimension p Chow group of Y and let $A^p(Y)$ be the subgroup in $CH^p(Y)$ generated by algebraically trivial algebraic cycles on Y. Suppose V is another algebraic variety over k with a closed point P_0. Let Z be an algebraic cycle of codimension p on the product $V \times Y$. For any closed point P on V we have the standard algebraic cycle $Z(P)$ of codimension n on Y. Obviously, the cycle

$$Z(P) - Z(P_0)$$

is algebraically trivial on Y. Then we obtain a map

$$V \to A^p(Y)$$

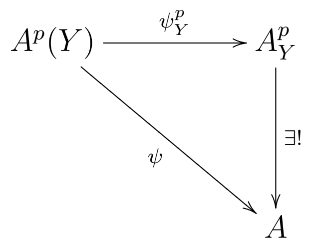
$$P \mapsto [Z(P) - Z(P_0)]$$

on Y. This map is nothing but the algebraic family of codimension n algebraically trivial cycle classes on Y determined by the algebraic cycle Z on $V \times Y$ and the fixed point P_0 on the parameter variety V.

If now A is an abelian variety over k, then a group homomorphism $A^p(Y) \to A$ is called to be regular if its pre-composition with any family of algebraic cycles $V \to A^p(Y)$ in the above sense is a regular morphism over k. A regular homomorphism

$$\psi_Y^p : A^p(Y) \to A_Y^p$$

into an abelian variety A_Y^p over k is called to be universal if, having another regular homomorphism $\psi : A^p(Y) \to A$, there exists a unique homomorphism of abelian varieties $A_Y^p \to A$, such that the diagram



commutes.

For example, in codimension $n = 1$, if Y is projective and nonsingular, the universal regular homomorphism is the isomorphism

$$A^1(Y) \stackrel{\sim}{\to} \mathrm{Pic}^0(Y) \; .$$

The main result in [25] says that ψ_Y^p always exists in codimension $p = 2$, for a nonsingular projective Y.

Let Y and X be two nonsingular projective varieties over k, and let Z be a correspondence of degree e from Y to X, where

$$e = \dim(X) - \dim(Y) \; .$$

In other words, Z is an algebraic cycle of codimension $\dim(X)$ on $Y \times X$. The correspondence Z induces a homomorphism Z_* from $CH^p(Y)$ to $CH^{p+e}(X)$ depending only on the class of rational equivalence of Z and preserving any adequate equivalence relation on algebraic cycles. In particular, one has the homomorphism

$$Z_* : A^p(Y) \to A^{p+e}(X) \; .$$

Then, for any regular morphism $\phi : A^{p+e}(X) \to B$ the composition $\phi \circ Z_* : A^p(Y) \to B$ is regular. If ψ_X^p exists for Y then $\phi \circ Z_*$ induces the corresponding regular homomorphism from A_Y^p to B. If Z is the graph of a regular morphism $r : Y \to X$ then we will write r_* instead of Z_*.

Before to go any further, we need to prove a few fairly simple lemmas.

Lemma 4.2.1. *Let V be an irreducible quasi-projective algebraic variety over an uncountable algebraically closed field. Then V cannot be written as a countable union of its Zariski closed subsets, each of which is not the whole V.*

Proof. Since V is supposed to be irreducible, without loss of generality we may assume that V is affine. Let d be the dimension of V and suppose $V = \cup_{n\in\mathbb{N}} V_n$ is the union of closed subsets V_n in V, such that $V_n \neq V$ for each n. By Emmy Noether's lemma, there exists a finite surjective morphism $f : V \to \mathbb{A}^d$ over k. Let W_n be the image of V_n under f. Since f is finite, it is proper. Therefore, W_n are closed in $\mathbb{A}^d$, so that we obtain that the affine space $\mathbb{A}^d$ is the union of W_n's. Since the ground field is uncountable, the set of all hyperplanes in $\mathbb{A}^d$ is uncountable. Therefore, there exists a hyperplane H, such that $W_n \not\subset H$ for any index n. Induction reduces the assertion of the lemma to the case when $d = 1$. □

A countable union $V = \cup_{n\in\mathbb{N}} V_n$ of algebraic varieties will be called irredundant if V_n is irreducible for each n and $V_m \not\subset V_n$ for $m \neq n$. In an irredundant decomposition, the sets V_n will be called c-components of V.

Lemma 4.2.2. *Let Z be a countable union of algebraic varieties over an uncountable algebraically closed ground field. Then Z admits an irredundant decomposition, and such an irredundant decomposition is unique.*

Proof. Let $Z = \cup_{n\in\mathbb{N}} Z'_n$ be a countable union of algebraic varieties over k. For each n let $Z'_n = Z'_{n,1} \cup \cdots \cup Z'_{n,r_n}$ be the irreducible components of Z'_n. Ignoring all components $Z'_{m,i}$ with $Z'_{m,i} \subset Z'_{n,j}$ for some n and j we end up with a irredundant decomposition. Having two irredundant decompositions $Z = \cup_{n\in\mathbb{N}} Z_n$ and $Z = \cup_{n\in\mathbb{N}} W_n$, suppose there exists Z_m such that Z_m is not contained in W_n for any n. Then Z_m is the union of the closed subsets $Z_m \cap W_n$, each of which is not Z_m. This contradicts to Lemma 4.2.1. Therefore, any Z_m is contained in some W_n. By symmetry, any W_n is in Z_l for some l. Then $Z_m \subset Z_l$. By irredundancy, $l = m$ and $Z_m = W_n$. □

Lemma 4.2.3. *Let A be an abelian variety over an uncountable algebraically closed field, and let K be a subgroup which can be represented as a countable union of Zariski closed subsets in A. Then the irredundant decomposition of K contains a unique irreducible component passing through* 0*, and this component is an abelian subvariety in A.*

Proof. Let $K = \cup_{n\in\mathbb{N}} K_n$ be the irredundant decomposition of K, which exists by Lemma 4.2.2. Since $0 \in K$, there exists at least one component in the irredundant decomposition, which contains 0. Suppose there are s components $K_1, \ldots, K_s$ containing 0 and $s > 1$. The summation in K gives the regular morphism from the product $K_1 \times \cdots \times K_s$ into A, whose image is the irreducible Zariski closed subset $K_1 + \cdots + K_s$ in A. By Lemma 4.2.1, there exists $n \in \mathbb{N}$, such that

$K_1 + \cdots + K_s \subset K_n$ and so $K_i \subset K_n$ for each $1 \leq i \leq n$. By irredundancy, $s = 1$, which contradicts to the assumption $s > 1$.

After renumbering of the components, we may assume that $0 \in K_0$. If $K_0 = \{0\}$, then K_0 is trivially an abelian variety. Suppose $K_0 \neq \{0\}$ and take a non-trivial element x in K_0. Since $-x + K_0$ is irreducible, it must be in some K_n by Lemma 4.2.1. As $0 \in -x + K_0$ it follows that $0 \in K_n$ and so $n = 0$. It follows that $-x \in K_0$. Similarly, since $K_0 + K_0$ is irreducible and contains K_0, we see that $K_0 + K_0 = K_0$. Being a Zariski closed abelian subgroup in A, the set K_0 is an abelian subvariety in A. □

Let now $r : Y \to X$ be a proper morphism of nonsingular projective varieties over k and impose the following two assumptions on the variety Y, which will be considered as standard assumptions throughout the paper.

(A) the universal regular homomorphism ψ^p_Y exists and is an isomorphism of abelian groups and

(B) the quotient group of algebraic cycles of codimension p on Y modulo algebraically trivial algebraic cycles is $\mathbb{Z}$.

Let

$$r_* : A^p(Y) \to A^{p+e}(X)$$

be the push-forward homomorphism induced by the proper morphism r, where

$$e = \dim(X) - \dim(Y) .$$

Let also

$$K = K^p_r$$

be the image of the kernel of the homomorphism r_* under the isomorphism ψ^p_Y between $A^p(Y)$ and A^p_Y.

Proposition 4.2.4. *Under the above assumptions, there exists an abelian subvariety A_0 and a countable subset Ξ in $A = A^p_Y$, such that K is the union of the shifts of the abelian subvariety A_0 by the elements from Ξ in A.*

Proof. Fix a polarization $Y \subset \mathbb{P}^m$. For any two closed points A and B in $C^p_d(Y)$ the difference $A - B$ is a codimension p cycle of degree zero on Y. The assumption that $CH^p(Y)$ splits into $A^p(Y)$ and $\mathbb{Z}$ guarantees that $A - B$ is algebraically equivalent to zero on Y. Then the map θ^p_d from $C^p_{d,d}(Y)$ to $CH^p(Y)$, defined in Section 4.1, takes its values in $A^p(Y)$, and we obtain the map θ^p_d from $C^p_{d,d}(Y)$ to $A^p(Y)$.

Since ψ^p_Y is an isomorphism, it follows that $A^p(Y)$ is weakly representable. In other words, there exists a nonsingular projective curve C and an algebraic cycle Z of codimension p on $C \times Y$, such that the homomorphism

$$[Z]_* : A^1(C) \to A^p(Y) ,$$

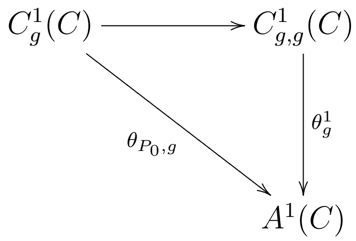
induced by the cycle class $[Z]$, is onto. The Chow scheme $C^1_d(C)$ is nothing but the d-th symmetric power of the curve C. Then we have the map θ^1_d from the 2-fold product $C^1_{d,d}(C)$ of this symmetric power to $A^1(C)$.

If g is the genus of the curve C, then the map θ^1_g is surjective. Indeed, fix some point P_0 on C and consider the map

$$\theta_{P_0,g} : C^1_g(C) \to A^1(C)$$

$$\sum_{i=1}^{g} P_i \mapsto \sum_{i=1}^{g} [P_i - P_0]$$

where for a point P in C, $[P - P_0]$ denote its class in $A^1(C)$.Now we have the following commutative diagram.



Now we prove that the map $\theta_{P_0,g}$ is onto. So for any z in $A^1(C)$, we write z as $[D]$, where D is a divisor on C representing z. Then $\theta^{-1}_{P_0,g}(z) = \theta^{-1}_{P_0,g}([D])$ is the set of all effective divisors $\sum_{i=1}^{g} P_i$ of degree g on $A^1(C)$ such that

$$\sum_{i=1}^{g} P_i - gP_0 - D = \mathrm{div}(f)$$

for some nonzero f in $k(C)$. This means that

$$\sum_{i=1}^{g} P_i = (gP_0 + D) + \mathrm{div}(f)\ .$$

So this gives rise to an element f in $\mathscr{L}(D + gP_0)$. On the other hand for any f in $\mathscr{L}(D + gP_0)$ we have that

$$(gP_0 + D) + \mathrm{div}(f)$$

is effective, write it as

$$\sum_{i=1}^{n} P_i$$

and we have

$$\sum_{i=1}^{n} P_i - gP_0 - D = \mathrm{div}(f)$$

therefore $\sum_{i=1}^{n} P_i - gP_0$ is rationally equivalent to D on C. Since D is of degree zero and rational equivalence preserves degree we get that

$$n = g \ .$$

This gives us that $\sum_{i=1}^{g} P_i$ is in the inverse image $\theta_{P_0,g}^{-1}([D])$. So we get that

$$\theta_{P_0,g}^{-1}([D]) \cong \mathscr{L}(D + gP_0) \ .$$

Now by the Riemann-Roch theorem we get that

$$l(D + gP_0) - l(K_C - (D + gP_0)) = \deg(D + gP_0) - g + 1$$

where $l(D)$ is the dimension of the vector space $\mathscr{L}(D)$, for a divisor D on C and K_C is the canonical divisor on C. Since $\deg(D + gP_0) = g$, we get that

$$l(D + gP_0) - l(K_C - (D + gP_0)) = 1$$

so

$$l(D + gP_0) = l(K_C - (D + gP_0)) + 1 \ .$$

So we have the fiber $\theta_{P_0,g}^{-1}([D])$ is non-empty. So the map $\theta_{P_0,g}$ is surjective. Therefore it follows that θ_g^1 is surjective.

For any two closed points P and P' on C the degree of the cycle classes ${\rm pr}_{Y*}({\rm pr}_C^*[P] \cdot [Z])$ and ${\rm pr}_{Y*}({\rm pr}_C^*[P'] \cdot [Z])$ is the same. Indeed, for that we observe that

$${\rm pr}_{Y*}({\rm pr}_C^*[P] \cdot Z) = Z(P)$$

and that for any closed points P and P' the cycle $Z(P) - Z(P')$ is algebraically equivalent to zero. In terms of Chow varieties we can say that two algebraic cycles A and B of the same codimension are algebraically equivalent if there exist two points x_0 and x_1 on C, positive cycle D on C and a regular morphism f from C to the Chow scheme $C^p(Y)$ of Y, such that

$$f(x_0) = A + D \quad \text{and} \quad f(x_1) = B + D \ ,$$

see Theorem 3 in [30]. We know that $C^p(Y)$ is equal to $\coprod_i C_i^p(Y)$, where $C_i^p(Y)$ is the Chow scheme parametrizing the codimension p degree i algebraic cycles on Y. Since C is connected and f is regular we have that the image of f is contained in one of $C_i^p(Y)$. Therefore we get

$$\deg(A + D) = \deg(B + D) \ ,$$

which gives that

$$\deg(A) = \deg(B) \ .$$

Since $Z(P)$ and $Z(P')$ are algebraically equivalent we have that

$$\deg(Z(P)) = \deg(Z(P')) \ .$$

Let d_0 be the degree of the cycle class $\mathrm{pr}_{Y*}(\mathrm{pr}_C^*[P]\cdot[Z])$, for a closed point P on C, and set d to be the product of d_0 with the genus g. Then the surjectivity of the map θ_g^1 from $C_{g,g}^1(C)$ to $A^1(C)$ and the homomorphism $[Z]_*$ from $A^1(C)$ to $A^p(Y)$ give the surjectivity of the map θ_d^p from $C_{d,d}^p(Y)$ to $A^p(Y)$. Indeed, take an element a in $A^p(Y)$. Since the homomorphism $[Z]_*$ is surjective, there exists b in $A^1(C)$ such that $[Z]_*(b)=a$. Now the map θ_g^1 from $C_{g,g}^1(C)$ to $A^1(C)$ is surjective, that gives us that b can be written as

$$\theta_g^1\left(\sum_i P_i,\sum_j Q_j\right)\ .$$

Therefore,

$$[Z]_*(b)=[Z]_*\left(\sum_i[P_i]-\sum_j[Q_j]\right)$$

where $[P]$ denotes the cycles class in the group $A^1(C)$ determined by the point P. By definition of $[Z]_*$ the above is

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(P_i))]-\sum_j[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(Q_j))]\ .$$

The degree of $\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(P_i))$ is same as the degree of $\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(Q_j))$. Therefore, in the above summation, we get that the degree of

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(P_i))]$$

is same as the degree of

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(Q_j))]\ .$$

Now write

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(P_i))]$$

as $[A]-[B]$ and

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(Q_j))]$$

as $[C]-[D]$. Then we have

$$\sum_i[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(P_i))]-\sum_j[\mathrm{pr}_{Y*}(Z\cdot\mathrm{pr}_C^*(Q_j))]$$

is equal to

$$[A]-[B]-[C]+[D]=[A]+[D]-([B]+[C])\ ,$$

and by the above degree reason we have

$$\deg(A+D)=\deg(B+C)\ .$$

Thus, we have the point $(A+D, B+C)$ on $C^p_{d,d}(Y)$ for some d. This proves that the map θ^p_d is surjective from $C^p_{d,d}(Y)$ to $A^p(Y)$ for some large enough d.

Let K be the kernel of the push-forward homomorphism r_* from $A^p(Y)$ to $A^{p+e}(X)$. Clearly, K is also the kernel of the push-forward homomorphism r_* from $A^p(Y)$ to $CH^{p+e}(X)$. Consider the commutative square

$$\begin{array}{ccc} C^p_{d,d}(Y) & \xrightarrow{r_*} & C^{p+e}_{d,d}(X) \\ \Big\downarrow{\scriptstyle \theta^p_d} & & \Big\downarrow{\scriptstyle \theta^{p+e}_d} \\ A^p(Y) & \xrightarrow{r_*} & CH^{p+e}(X) \end{array}$$

where the top homomorphism r_* is induced by the proper morphism r. Let K' be the pre-image of 0 under the composition

$$\theta^{p+e}_d \circ r_* \circ Z_*$$

from $C^1_{d,d}$ to $A^{p+e}(X)$. Since the homomorphisms

$$\theta^1_d : C^1_{d,d}(C) \to A^1(C)$$

and

$$Z_* : A^1(C) \to A^p(Y)$$

are both surjective, it follows that the kernel of the push-forward homomorphism

$$r_* : A^p(Y) \to A^{p+e}(X)$$

is the image of K' under the composition

$$Z_* \circ \theta^1_d : C^1_{d,d}(C) \to A^p(Y) \; .$$

Respectively, $K = K^p_r$ is the image of K' under the triple composition

$$C^1_{d,d}(C) \stackrel{\theta^1_d}{\to} A^1(C) \stackrel{\psi^1_C}{\to} A^1_C \stackrel{Z_*}{\to} A^p_Y$$

By Corollary 4.1.2, $(\theta^{p+e}_d)^{-1}(0)$ is the union of a countable collection of closed subvarieties in $C^{p+e}_{d,d}(X)$. Then so is K'. As the morphisms in the latter triple composition are all regular, K is again the union of a countable collection of closed subvarieties in $C^1_{d,d}(C)$. By Lemma 4.2.2, the set K admits a unique irredundant decomposition. Let A_0 be the unique component of that decomposition passing through 0, which is an abelian subvariety in the abelian variety $A = A^p_Y$ by Lemma 4.2.3. Let us show that A_0 is the required abelien variety $A^p_{Y,r}$ from the statement of the proposition.

Indeed, for any x in K the set $x + A_0$ is an irreducible Zariski closed subset in K. Since K is then union

$$\cup_{x \in K}(x + A_0) \ ,$$

ignoring each set $x + A_0$ which is a subsets in $y + A_0$ for some $y \in K$, we can find a subset $\Xi = \Xi_r^p$ in K, such that

$$K = \cup_{x \in \Xi}(x + A_0)$$

and for any two elements $x, x' \in \Xi$ the irreducible sets $x + A_0$ and $x' + A_0$ are not contained one in another. Take the irredundant decomposition

$$K = \cup_{n \in \mathbb{N}} K_n \ .$$

It exists by Lemma 4.2.2. Since $x + A_0$ is irreducible, it is contained in K_n for some n by Lemma 4.2.1. Then $A_0 \subset -x + K_n$. Similarly, $-x + K_n \subset K_l$ for some l, so that $K_l = A_0$ by Lemma 4.2.3. This yields $x + A_0 = K_n$, for each $x \in \Xi$. It means that the set $\Xi = \Xi_r^p$ is countable. $\square$

4.3 The Gysin homomorphism on étale cohomology groups

For short, let $A = A_Y^p$, $K = K_r^p$ and let A_0 be as in the proof of Proposition 4.2.4. Choose an ample line bundle L on the abelian variety A. Let

$$i : A_0 \to A$$

be the closed embedding of A_0 into A, and let L_0 be the pull-back of L to A_0 under the closed embedding i. Define the homomorphism ζ on divisors via the commutative diagram

$$\begin{array}{ccc} A^1(A_0) & \xrightarrow{\ \zeta\ } & A^1(A) \\ \downarrow{\scriptstyle (\lambda_{L_0})_*} & & \uparrow{\scriptstyle \lambda_L^*} \\ A^1(A_0^\vee) & \xrightarrow{\ i^{\vee *}\ } & A^1(A^\vee) \end{array}$$

Similarly, we define the homomorphism $\zeta_{\mathbb{Z}_l}$ on cohomology by means of the commutative diagram

$$\begin{array}{ccc} H^1_{ét}(A_0, \mathbb{Z}_l) & \xrightarrow{\ \zeta_{\mathbb{Z}_l}\ } & H^1_{ét}(A, \mathbb{Z}_l) \\ \downarrow{\scriptstyle \lambda_{L_0 *}} & & \uparrow{\scriptstyle \lambda_L^*} \\ H^1_{ét}(A_0^\vee, \mathbb{Z}_l) & \xrightarrow{\ i^{\vee *}\ } & H^1_{ét}(A^\vee, \mathbb{Z}_l) \end{array}$$

and analogously for the homomorphism

$$\zeta_{\mathbb{Q}_l/\mathbb{Z}_l} : H^1_{ét}(A_0, \mathbb{Q}_l/\mathbb{Z}_l) \to H^1_{ét}(A_0, \mathbb{Q}_l/\mathbb{Z}_l) \ .$$

The homomorphism $\zeta_{\mathbb{Z}_l}$ induces the injective homomorphisms

$$\zeta_{\mathbb{Q}_l} : H^1_{ét}(A_0, \mathbb{Q}_l) = H^1_{ét}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l \to H^1_{ét}(A_0, \mathbb{Q}_l) = H^1_{ét}(A, \mathbb{Z}_l) \otimes \mathbb{Q}_l$$

and

$$\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l : H^1_{ét}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^1_{ét}(A_0, \mathbb{Z}_l) \otimes \mathbb{Q}_l/\mathbb{Z}_l \ .$$

Next, since ψ^p_Y is an isomorphism, the group $A^p(Y)$ is weakly representable. It means that there exists a smooth projective curve Γ, a cycle Z of codimension p on $\Gamma \times Y$, and an algebraic subgroup $G \subset J_\Gamma$ in the Jacobian variety J_Γ, such that the induced homomorphism

$$z_* : J_\Gamma = A^1(\Gamma) \to A^p(Y) \simeq A$$

is surjective, and its kernel is the group G. Here z is the cycle class of Z in the Chow group $CH^p(\Gamma \times Y)$. Furthermore, the class z gives the morphism

$$z : M(\Gamma) \otimes \mathbb{L}^{p-1} \to M(Y) \ ,$$

where $M(-)$ is the functor from nonsingular projective varieties over k to (contravariant) Chow motives over k, $\mathbb{L}$ is the Lefschetz motive and $\mathbb{L}^n$ is the n-fold tensor power of $\mathbb{L}$. Then we just copy the construction from [14].

Namely, fix a point on the curve Γ and consider the induced embedding

$$i_\Gamma : \Gamma \to J_\Gamma \ .$$

Let

$$\alpha : J_\Gamma \to A$$

be the projection from the Jacobian J_Γ onto the abelian variety

$$A = J_\Gamma / G \ .$$

Let

$$w = z \circ (M(\alpha \circ i_\Gamma) \otimes \mathrm{id}_{\mathbb{L}^{p-1}})$$

be the composition in the category of Chow motives with coefficients in $\mathbb{Z}$. Then w is a morphism

$$w : M(A) \otimes \mathbb{L}^{p-1} \to M(Y) \ ,$$

and it induces the homomorphism

$$w_* : H^1_{ét}(A, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$$

on the l-adic Weil cohomology groups.

Proposition 4.3.1. *The image of the composition*

$$H^1_{ét}(A_0, \mathbb{Q}_l(1-p)) \stackrel{\zeta_{\mathbb{Q}_l}}{\to} H^1_{ét}(A, \mathbb{Q}_l(1-p)) \stackrel{w_*}{\to} H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$$

is contained in the kernel of the Gysin homomorphism

$$H^{2p-1}_{ét}(Y, \mathbb{Q}_l) \stackrel{r_*}{\to} H^{2(p+e)-1}_{ét}(X, \mathbb{Q}_l) .$$

Proof. For the proof we will be using Bloch's l-adic Abel-Jacobi maps. For any abelian group A, a prime l and positive integer n let A_{l^n} be the kernel of the multiplication by l^n endomorphism of A and let $A(l)$ be the l-primary part of A, i.e. the union of the groups A_{l^n} for all n. For any smooth projective variety V over k, there is a canonical homomorphism

$$\lambda^p_l(V) : CH^p(V)(l) \to H^{2p-1}_{ét}(V, \mathbb{Q}_l/\mathbb{Z}_l(p)) ,$$

constructed by Bloch in [6]. The homomorphisms $\lambda^p_l(V)$ are functorial with respect to the action of correspondences between smooth projective varieties over k. Moreover, the homomorphisms

$$\lambda^1_l(V) : CH^1(V)(l) \to H^1_{ét}(V, \mathbb{Q}_l/\mathbb{Z}_l(1))$$

are isomorphisms, loc.cit.

Now, we have the following commutative diagram

$$\begin{array}{ccc} A_0(l) & \stackrel{i}{\longrightarrow} & A(l) \\ \downarrow & & \downarrow \\ A^1(A_0)(l) & \stackrel{\zeta}{\longrightarrow} & A^1(A)(l) \end{array}$$

Since A_0 and A are abelian varieties, their Néron-Severi groups are torsion free. It follows that

$$CH^1(A_0)(l) = A^1(A_0)(l)$$

and

$$CH^1(A)(l) = A^1(A)(l) ,$$

so that we actually have the isomorphism

$$\lambda^1_l(A_0) : A^1(A_0)(l) \stackrel{\sim}{\to} H^1_{ét}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) ,$$

and the isomorphism

$$\lambda^1_l(A) : A^1(A)(l) \stackrel{\sim}{\to} H^1_{ét}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) .$$

Similarly, one has the isomorphisms $\lambda_l^1(A_0^\vee)$ and $\lambda_l^1(A^\vee)$ for the dual abelian varieties.

The functorial properties of Bloch's maps λ_l^1 give that the diagram

$$\begin{array}{ccc} A^1(A_0)(l) & \xrightarrow{\zeta} & A^1(A)(l) \\ {\scriptstyle \lambda_l^1(A_0)}\big\downarrow \wr & & {\scriptstyle \lambda_l^1(A)}\big\downarrow \wr \\ H^1_{\acute{e}t}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \end{array} \tag{4.1}$$

is commutative.

Since the Bloch's Abel Jacobi maps are functorial with respect to the action of correspondences between smooth projective varieties over k we get that the diagrams

$$\begin{array}{ccc} A^1(A)(l) & \xrightarrow{\omega_*} & A^p(Y)(l) \\ {\scriptstyle \lambda_l^1(A)}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_Y^p(l)} \\ H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\omega_*} & H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) \end{array}$$

and

$$\begin{array}{ccc} A^p(Y)(l) & \xrightarrow{r_*} & A^{p+e}(X)(l) \\ {\scriptstyle \lambda_l^p(A)}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_X^{p+e}(l)} \\ H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) & \xrightarrow{r_*} & H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(p+e)) \end{array}$$

are commutative. They help to observe that the composition $r_* \circ \omega_* \circ \zeta$ is zero, because the abelian variety A sits inside the kernel of the push-forward homomorphism r_*. Therefore it follows that

$$r_* \circ \omega_* \circ \zeta_{\mathbb{Q}_l/\mathbb{Z}_l} = 0$$

since $\lambda_l^1(A_0)$ is an isomorphism.

For a smooth projective V over k, one has the homomorphisms

$$\varrho_l^{i,j}(V) : H^i_{\acute{e}t}(V, \mathbb{Z}_l(j)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^i_{\acute{e}t}(V, \mathbb{Q}_l/\mathbb{Z}_l(j)) \ ,$$

with finite kernels and cokernels, considered in [14]. In particular, we have the commutative diagram

$$\begin{array}{ccc} H^1_{\acute{e}t}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \\ \uparrow {\scriptstyle \varrho_l^{1,1}(A_0)} & & \uparrow {\scriptstyle \varrho_l^{1,1}(A)} \\ H^1_{\acute{e}t}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l} & H^1_{\acute{e}t}(A, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \end{array} \tag{4.2}$$

Let

$$\sigma : A_0 \stackrel{\sim}{\to} A^1(A_0^\vee)$$

and

$$\sigma : A \stackrel{\sim}{\to} A^1(A^\vee)$$

be the autoduality isomorphisms. The above morphism of motives w from $M(A) \otimes \mathbb{L}^{p-1}$ to $M(Y)$ induces the homomorphism

$$w_* : A^1(A) \to A^p(Y)$$

on Chow groups. A straightforward verification shows that the diagram

$$\begin{array}{ccc} A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\ \uparrow {\scriptstyle \lambda_L^*} & & \uparrow {\scriptstyle (\psi_Y^p)^{-1}} \\ A^1(A^\vee)(l) & \xleftarrow{\sigma} & A(l) \end{array} \tag{4.3}$$

is commutative. The homomorphism w_* on Chow groups and the homomorphism

$$w_* : H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p))$$

induced by w on cohomology fit into the commutative diagram

$$\begin{array}{ccc} A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\ {\scriptstyle \lambda_l^1(A)} \downarrow {\scriptstyle \sim} & & \downarrow {\scriptstyle \lambda_l^p(Y)} \\ H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{w_*} & H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) \end{array} \tag{4.4}$$

The commutativity of the diagrams (4.1), (4.2), (4.3) and (4.4), the definition of the abelian variety A_0 and easy diagram chase over the obvious commutative diagrams

$$\begin{array}{ccc} H^1_{\acute{e}t}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\omega_*} & H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) \\ \uparrow {\scriptstyle \varrho_l^{1,1}(A)} & & \uparrow {\scriptstyle \varrho_l^{2p-1,p}(Y)} \\ H^1_{\acute{e}t}(A, \mathbb{Z}_l(1)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\omega_*} & H^{2p-1}_{\acute{e}t}(Y, \mathbb{Z}_l(1)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \end{array}$$

and

$$\begin{CD}
H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p)) @>{r_*}>> H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(p+e)) \\
@A{\varrho_l^{2p-1,p}(A)}AA @A{\varrho_l^{2(p+e)-1,p+e}(X)}AA \\
H^{2p-1}_{\acute{e}t}(Y, \mathbb{Z}_l(p)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l @>{r_*}>> H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Z}_l(p+e)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l
\end{CD}$$

show that, since $r_* \circ \omega_* \circ \zeta_{\mathbb{Q}_l/\mathbb{Z}_l} = 0$, the image of the triple composition

$$r_* \circ w_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l) : H^1_{\acute{e}t}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Z}_l(p+e)) \otimes \mathbb{Q}_l/\mathbb{Z}_l$$

is contained in the kernel of the homomorphism

$$\varrho_l^{2(p+e)-1,p+e}(X) : H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Z}_l(p+e)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Q}_l/\mathbb{Z}_l(p+e)) \ ,$$

that is

$$\varrho_l^{2(p+e)-1,p+e} \circ r_* \circ \omega_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l) = 0 \ .$$

By [14] we know that the kernel of $\varrho^{2(p+e)-1,p+e}$ is finite therefore we have that image of $r_* \circ \omega_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l)$ is finite. But since it is a finitely generated $\mathbb{Z}_l$-module, we get that

$$r_* \circ \omega_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l) = 0 \ .$$

Finally, look at the commutative diagrams

$$\begin{CD}
H^1_{\acute{e}t}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l @>{\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l}>> H^1_{\acute{e}t}(A, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \\
@AAA @AAA \\
H^1_{\acute{e}t}(A_0, \mathbb{Q}_l(p)) @>{\zeta_{\mathbb{Q}_l}}>> H^1_{\acute{e}t}(A, \mathbb{Q}_l(1))
\end{CD}$$

$$\begin{CD}
H^1_{\acute{e}t}(A, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l @>{\omega_*}>> H^{2p-1}_{\acute{e}t}(Y, \mathbb{Z}_l(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \\
@AAA @AAA \\
H^1_{\acute{e}t}(A, \mathbb{Q}_l(1)) @>{\omega_*}>> H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l(p))
\end{CD}$$

$$\begin{CD}
H^{2p-1}_{\acute{e}t}(Y, \mathbb{Z}_l(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l @>{r_*}>> H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Z}_l(p+e)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \\
@AAA @AAA \\
H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l(p)) @>{r_*}>> H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Q}_l(p+e))
\end{CD}$$

Since the étale cohomology groups of smooth projective varieties with $\mathbb{Z}_l$-coefficients are finitely generated $\mathbb{Z}_l$-modules, it follows that the image of the triple composition $r_* \circ w_* \circ \zeta_{\mathbb{Q}_l}$ from $H^1_{\acute{e}t}(A_0, \mathbb{Q}_l(1))$ to $H^{2(p+e)-1}_{\acute{e}t}(X, \mathbb{Q}_l(p+e))$ is zero, which finishes the proof of the proposition. □

Remark 4.3.2. If $\dim(Y) = 1$ and $p = 1$, then $w_* : H^1_{\acute{e}t}(A, \mathbb{Q}_l(1-p)) \to H^1_{\acute{e}t}(Y, \mathbb{Q}_l)$ is an isomorphism by the standard argument. If $\dim(Y) = 3$ and $p = 2$, the homomorphism w_* between $H^1_{\acute{e}t}(A, \mathbb{Q}_l(-1))$ and $H^3_{\acute{e}t}(Y, \mathbb{Q}_l)$ is an isomorphism by Lemma 4.3 in [14].

In the applications below we will be dealing with the case when X is embedded into a projective space, the dimension of X is $2p$ and Y is a nonsingular hyperplane section of X, so that

$$\dim(Y) = 2p - 1 \qquad \text{and} \qquad e = 1 \ .$$

Assume also that $p = 1$ or 2, in order to have that w_* is an isomorphism by Remark 4.3.2.

If the group $H^{2p+1}_{\acute{e}t}(X, \mathbb{Q}_l)$ vanishes, then the primitive cohomology group coincides with $H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l)$. This is the case when X is a hypersurface in $\mathbb{P}^{2p+1}$. If $H^{2p+1}_{\acute{e}t}(X, \mathbb{Q}_l)$ is nonzero, then we need to construct an abelian subvariety A_1 in $A = A^p_Y$, such that the image of the injective homomorphism

$$H^1_{\acute{e}t}(A_1, \bar{\mathbb{Q}}_l) \to H^1_{\acute{e}t}(A, \bar{\mathbb{Q}}_l) \ ,$$

induced by the inclusion $A_1 \subset A$, would coincide with the kernel of the composition of the isomorphism w_* with the Gysin homomorphism r_* from $H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l)$ to $H^{2p+1}_{\acute{e}t}(X, \mathbb{Q}_l)$. Notice that the Gysin homomorphism is surjective by the Lefschetz hyperplane section theorem.

If $p = 1$, then X is a surface and Y is a curve. Then A^1_Y can be identified with the Albanese variety of Y and $\psi^1_Y : A^1(Y) \to A^1_Y$ is the Abel-Jacobi isomorphism for the curve Y. The group $A^2(X)$ also admits a universal regular homomorphism ψ^2_X, which is nothing but the Albanese mapping from $A^2(X)$ to the Albenese variety A^2_X of X over k. Let A_1 has to be taken to be the connected component of the kernel of the induced homomorphism from A^1_Y to A^2_X.

When $p = 2$ the situation is a bit more difficult. Suppose first that k is $\mathbb{C}$. Then, for any algebraic variety V over $\mathbb{C}$ and any non-negative integer n the étale cohomology group $H^n_{\acute{e}t}(V, \mathbb{Q}_l)$ is functorially isomorphic to the singular cohomology group

$$H^*(V(\mathbb{C}), \mathbb{Q}_l) = H^*(V(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l \ .$$

The étale cohomology groups with coefficients in $\mathbb{Q}_l$ can be further tensored with the algebraic closure $\bar{\mathbb{Q}}_l$ of the l-adic field over $\mathbb{Q}_l$. Fixing an isomorphism between

$\bar{\mathbb{Q}}_l$ and $\mathbb{C}$, the étale cohomology groups $H^*_{ét}(-, \bar{\mathbb{Q}}_l)$ are functorially isomorphic to the singular cohomology groups $H^*(-, \mathbb{C})$. The Gysin homomorphism r_* from $H^{2p-1}(Y, \mathbb{C})$ to $H^{2p+1}(X, \mathbb{C})$ is a morphism of Hodge structures, so that its kernel H_1 is a Hodge substructure in $H^{2p-1}(Y, \mathbb{C})$. Suppose $p = 2$. By Remark 4.3.2 the group $H^3(Y, \mathbb{C})$ is isomorphic to the group $H^1(A, \mathbb{C})$ via the homomorphism w_*, and w_* is obviously a morphism of Hodge structures too. It follows that H_1 is of weight 1. This gives an abelian subvariety A_1 in $A = A^2_Y$, where $A^2_Y = J^2(Y)_{\mathrm{alg}}$ (see [25]). Using the above functorial isomorphisms between étale and complex cohomology groups, one can show that the image of the injective homomorphism from $H^1_{ét}(A_1, \mathbb{Q}_l)$ to $H^1_{ét}(A, \mathbb{Q}_l)$, induced by the inclusion $A_1 \subset A$, coincides with the kernel of the composition of the isomorphism

$$w_* : H^1_{ét}(A, \mathbb{Q}_l(1-p)) \xrightarrow{\sim} H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$$

with the surjective Gysin homomorphism

$$r_* : H^{2p-1}_{ét}(Y, \mathbb{Q}_l) \to H^{2p+1}_{ét}(X, \mathbb{Q}_l)\ .$$

Let now $p = 2$ and k be an arbitrary uncountable algebraically closed field of characteristic 0. In such a situation we still can construct A_1 using the fact that k has an infinite transcendence degree over $\mathbb{Q}$. Let L be the algebraic closure in k of the minimal field of definition of the varieties X, Y, Γ, A and A_0, as well as the above correspondence z from Γ to Y, so that all these varieties, the homomorphisms w_* and r_* all are coming from the corresponding models

$$X_L\ ,\ \ Y_L\ ,\ \ \Gamma_L\ ,\ \ A_L\ ,\ \ A_{0,L}\ ,\ \ w_{L*} \quad \text{and} \quad r_{L*}$$

over L. Fixing an embedding of L into $\mathbb{C}$, we can now extend scalars from L to $\mathbb{C}$ getting the varieties

$$(X_L)_{\mathbb{C}}\ ,\ \ (Y_L)_{\mathbb{C}}\ ,\ \ (\Gamma_L)_{\mathbb{C}}\ ,\ \ (A_L)_{\mathbb{C}} \quad \text{and} \quad (A_{0,L})_{\mathbb{C}}$$

over $\mathbb{C}$, an similarly for w_{L*} and r_{*L}. Working over $\mathbb{C}$ we now can construct the abelian subvariety $(A_1)_{\mathbb{C}}$ as above. It has a model over a field extension L' of L inside $\mathbb{C}$, whose transcendence degree over L is finite. Since the transcendence degree of k over $\mathbb{Q}$ is infinite, we can embed L' back into k over L getting the needed abelian variety A_1 over k.

Now again the image of the homomorphism

$$H^1_{ét}(A_1, \mathbb{Q}_l) \to H^1_{ét}(A, \mathbb{Q}_l)$$

coincides with the kernel of the composition of the homomorphism w_* from $H^1_{ét}(A, \mathbb{Q}_l(1-p))$ to $H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$ with the homomorphism r_* from $H^{2p-1}_{ét}(Y, \mathbb{Q}_l)$ to $H^{2p+1}_{ét}(X, \mathbb{Q}_l)$.

The technique used in the proof of Proposition 4.3.1 allows to show that, in all cases, A_0 is a subvariety in A_1.

And, certainly, if $H^{2p+1}_{ét}(X, \mathbb{Q}_l) = 0$, then we set $A_1 = A$.

4.4 Very general versus geometric generic fibre in a family

In this section, k is an algebraically closed field whose transcendental degree over the primary subfield is infinite. We will be using the following terminology. If V is an algebraic scheme over k, a (Zariski) c-closed subset in V is a union of a countable collection of Zariski closed irreducible subsets in V. A (Zariski) c-open subset in V is the complement to a c-closed subset in V. A property P of points in V holds for a very general point on V if there exists a c-open subset U in V, such that P holds for each closed point in U.

Let S be an integral affine scheme of finite type over k. Let $I(S)$ be the ideal in $k[x_1, \cdots, x_k]$ of S and $f_1, \cdots, f_n$ are the generators of the ideal. Since k is of characteristic zero, the prime subfield of k is $\mathbb{Q}$, then attach the coefficients of the polynomials $f_1, \cdots, f_n$ to $\mathbb{Q}$, this is a finite extension of $\mathbb{Q}$, which is a countable subfield of k. Then let S_0 be the affine integral scheme defined by the ideal generated by $f_1, \cdots, f_n$ in $k_0[x_1, \cdots, x_k]$, denote this ideal by $I(S_0)$. Since we have that

$$k_0[x_1, \cdots, x_k]/I(S_0) \otimes_{k_0} k = k[x_1, \cdots, x_k]/I(S)$$

we get that

$$S = S_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k) \ .$$

Now let Z be a closed subscheme of S_0, and let $i_Z : Z \to S_0$ be the closed embedding. Since Z is a closed subscheme in the affine schme S_0, it is defined by an ideal $\mathfrak{a}$ in $k_0[S_0]$, since the field k_0 is countable and $\mathfrak{a}$ is finitely generated, we have that there are only countably many ideals $\mathfrak{a}$ in $k_0[S_0]$. Therefore we have only countably many closed subschemes Z inside S_0. For each Z let U_Z be the complement $S_0 \smallsetminus \mathrm{im}(i_Z)$, $Z_k = Z \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$, $(U_Z)_k = U_Z \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$. Let $(i_Z)_k$ be the pullback of i_Z with respect to the extension k over k_0. Then $(U_Z)_k$ is the complement $S \smallsetminus \mathrm{im}((i_Z)_k)$. Let us consider

$$U = S \smallsetminus \cup_Z \mathrm{im}((i_Z)_k) = \cap_Z (U_Z)_k$$

where the union is taken over closed subschemes Z such that $\mathrm{im}((i_Z)_k)$ is not equal to S. Now we prove that this condition is equivalent to the condition that $\mathrm{im}(i_Z) \neq S_0$. Suppose that $\mathrm{im}((i_Z)_k) \neq S$, now if $\mathrm{im}(i_Z) = S_0$, that would immediately give us that $\mathrm{im}((i_Z)_k) = S$, that would be a contradiction. Now suppose the opposite, that $\mathrm{im}(i_Z) \neq S_0$, and suppose if possible that $\mathrm{im}((i_Z)_k) = S$. That would give us that

$$Z \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k) \cong S \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k) \ .$$

Then we have that

$$k_0[S_0]/\mathfrak{a} \otimes_{k_0} k = k_0[S_0] \otimes_{k_0} k \ .$$

That is we have

$$k[S]/\mathfrak{a} \otimes_{k_0} k = k[S]$$

where $\mathfrak{a} \otimes_{k_0} k$ denotes the ideal $\mathfrak{a}$ extended by the scalars in k. Since we have that

$$k[S]/\mathfrak{a} \otimes_{k_0} k = k[S]$$

we have that any prime ideal of $k[S]$ contains the ideal $\mathfrak{a} \times_{k_0} k$, since $k[S]$ is an integral domain, this in particular implies that $\{0\}$ contains the ideal $\mathfrak{a} \otimes_{k_0} k$. Therefore we have that

$$\mathfrak{a} \otimes_{k_0} k = \{0\} \ .$$

Since k is of characteristic zero it follows that $\mathfrak{a} = 0$, that is a contradiction to the fact that $\mathrm{im}(i_Z) \neq S$. Now the set U is the complement to the countable union of Zariski closed subsets, therefore it is c-open (see also the proof of Lemma 2.1 in [39]).

Proposition 4.4.1. *For any closed k-point P in U, one can construct a field isomorphism between $\overline{k(S)}$ and k, whose value at $f \in k_0[S_0]$ is $f(P)$.*

Proof.

Let P be a closed k-point in the above defined subset U in S, therefore there exists a morphism

$$f_P : \mathrm{Spec}(k) \to S$$

and its image under the projection

$$\pi : S \to S_0$$

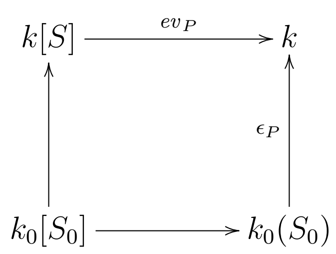
belong to U_Z for every closed subscheme Z of S_0. Therefore the image is the generic point η_0 of the scheme S_0, since the generic point of an integral scheme is unique. So there exists a morphism

$$h_P : \mathrm{Spec}(k) \to \mathrm{Spec}(k_0(S_0)) = \eta_0$$

such that we have

$$\pi \circ f_P = g_0 \circ h_P$$

where g_0 is a morphism from the generic point η_0 to S_0. In terms of commutative rings we then have the following commutative diagram. Let ev_P be the evaluation at P from $k[S]$ to k.



Here ϵ_P is the morphism at the level of commutative rings corresponding to the morphism

$$h_P : \mathrm{Spec}(k) \to \mathrm{Spec}(k_0(S_0)) .$$

Now we prove that the morphism

$$k_0[S_0] \to k[S]$$

is injective. Write $k_0[S_0]$ as $k[x_1, \cdots, x_k]/I(S_0)$ and $k[S]$ as $k[x_1, \cdots, x_k]/I(S)$. Therefore the homomorphism

$$k_0[S_0] \to k[S]$$

is given by

$$f(x_1, \cdots, x_k) + I(S_0) \mapsto f(x_1, \cdots, x_k) + I(S) .$$

Now suppose that $f(x_1, \cdots, x_k)+I(S) = 0$, that is $f(x_1, \cdots, x_k)$ belongs to $I(S)$, but at the same time we have that the coefficients of f are in k_0, therefore we get that $f(x_1, \cdots, x_k)$ is in $I(S_0)$, whence the homomorphism is injective. Therefore we get that $k_0[S_0] \smallsetminus \{0\}$ is a multiplicative system in $k[S]$. Now we check that the localization $(k_0[S_0] \smallsetminus \{0\})^{-1}k[S]$ is isomorphic to the tensor product

$$k[S] \otimes_{k_0[S_0]} k_0(S_0) .$$

Let us define the homomorphism

$$\Phi : (k_0[S_0] - \{0\})^{-1}k[S] \to k[S] \otimes_{k_0[S_0]} k_0(S_0)$$

as follows

$$\Phi(a/b) = a \otimes \frac{1}{b}$$

on the other hand we define the homomorphism

$$\Psi : k[S] \otimes_{k_0[S_0]} k_0(S_0) \to (k_0[S_0] \smallsetminus \{0\})^{-1}k[S]$$

as follows

$$\Psi(f \otimes g/h) = fg/h .$$

It is easy to check that the Φ and Ψ are inverses to each other. This is why there is a unique universal morphism rings ε_P from $k[S] \otimes_{k_0[S_0]} k_0(S_0)$ to k such that its restriction to $k[S]$ is ev_P and its restriction to $k_0(S_0)$ is ϵ_P. Our aim is now to construct an embedding of $k(S)$ into k whose restriction to $k_0(S_0)$ is ϵ_P. Let d be the dimension of S_0. By the Noether's normalization lemma there exists d algebraically independent elements $x_1, \cdots, x_d$ in $k_0[S_0]$ such that the latest ring is integral over $k_0[x_1, \cdots, x_d]$. Therefore it follows that $k_0(x_1, \cdots, x_d)$ is algebraic over $k_0(S_0)$. Then extending the scalars we get that $k[S]$ is integral over $k[x_1, \cdots, x_d]$ and $k(S)$ is algebraic over $k(x_1, \cdots, x_d)$. Let $b_i = ev_P(x_i)$ for

$i = 1, \cdots, d$. Since $P \in U$ we get that $b_1, \cdots, b_d$ are algebraically independent, because if they are algebraically dependent then there exists a polynomial f in d variables such that

$$f(b_1, \cdots, b_d) = 0$$

that will imply that $(b_1, \cdots, b_d)$ is in the closed subscheme of S_0 defined by f, that would be a contradiction. Now extend the set $\{b_1, \cdots, b_d\}$ to a transcendental basis B of k over k_0, now we can choose k_0 to be algebraically closed, in that case we have

$$k = k_0(B) \ .$$

Since B is of infinite cardinality so is the set $B \smallsetminus \{b_1, \cdots, b_d\}$. Choose and fix a bijection

$$\vartheta_{P,B} : B \to B \smallsetminus \{b_1, \cdots, b_d\}$$

it gives us a field embedding

$$\theta_{P,B} : k = k_0(B) \cong k_0(B \smallsetminus \{b_1, \cdots, b_d\}) \subset k_0(B)$$

over k_0 such that the set $\{b_1, \cdots, b_d\}$ is algebraically independent over $\theta_{P,B}(k)$. Therefore $\theta_{P,B}$ induces a field embedding

$$\theta_{P,B} : k(x_1, \cdots, x_d) \to k$$

by sending

$$x_i \mapsto b_i$$

for each i. By the commutativity of the above diagram it follows that the restriction of $\theta_{P,B}$ to $k_0(x_1, \cdots, x_d)$ is ϵ_P. Now we prove that $k(S)$ is the tensor product of $k(x_1, \cdots, x_d)$ and $k_0(S_0)$ over $k_0(x_1, \cdots, x_d)$. Let us define the homomorphism on simple tensors and then extend it linearly

$$\Phi : k(x_1, \cdots, x_d) \otimes_{k_0(x_1, \cdots, x_d)} k_0(S_0) \to k(S)$$

by

$$f(x_1, \cdots, x_d)/g(x_1, \cdots, x_d) \otimes a/b \mapsto af(x_1, \cdots, x_d)/bg(x_1, \cdots, x_d)$$

on the other hand let us consider $k(S)$, consider an element a/b, where a, b belongs to $k[S]$. Now it is enough to define the map on $k[S]$ and then extend it in such a way that it will be a homomorphism from $k(S)$ to the tensor product. For that let x be an element in $k[S]$ and write it as $\sum_i a_i y_i$ where $y_1, \cdots, y_d$ is a set of generators of $k[S]$ over $k[x_1, \cdots, x_d]$ as well as of $k_0[S_0]$ over $k_0[x_1, \cdots, x_d]$. Then send x to $\sum a_i \otimes y_i$. Now define the homomorphism

$$\Psi : k(S) \to k_0(S_0) \otimes_{k_0(x_1, \cdots, x_d)} k(x_1, \cdots, x_d)$$

by sending

$$x/y \mapsto \left(\sum_i a_i \otimes y_i)(\sum_j b_j \otimes y_j\right)^{-1}$$

where

$$x = \sum_i a_i y_i, \quad y = \sum_j b_j y_j \ .$$

Now the homomorphism Φ sends $\Psi(x/y)$ to $\sum_i a_i y_i / \sum b_j y_j$ which is x/y, this proves that $\Phi \circ \Psi = id$. Now we have to prove that $\Psi \circ \Phi$ is identity. For that start with a tensor v in the tensor product $k(x_1, \cdots, x_d) \times_{k_0(x_1,\cdots,x_d)} k_0(S_0)$ it will look like

$$\sum a_i/b_i \otimes f_i/g_i$$

this goes to

$$\sum a_i f_i / b_i g_i$$

by Φ. Now we apply Ψ we get that

$$\Psi(a_i f_i/b_i g_i) = \Psi\left(a_i\left(\sum_j c_{ij} y_{ij}\right)/b_i\left(\sum_k d_{ij} y_{ij}\right)\right)$$

that is equal to

$$\left(\sum a_i \otimes c_{ij} y_{ij}\right) \cdot \left(\sum b_i \otimes d_{ij}^{-1} y_{ij}\right)^{-1}$$

this is same as $a_i/b_i \otimes f_i/g_i$ so we get that

$$\Psi \circ \Phi = id \ .$$

Therefore we get a uniquely defined field embedding $\Theta_{P,B}$ from $k(S)$ to k, which can be extended to an isomorphism $e_P : k(\bar{S}) \to k$, by the above commutative diagram we have that

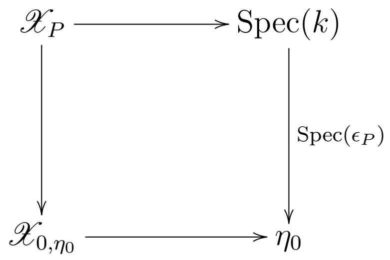
$$e_P(f) = f(P)$$

for each f in $k_0[S_0]$. □

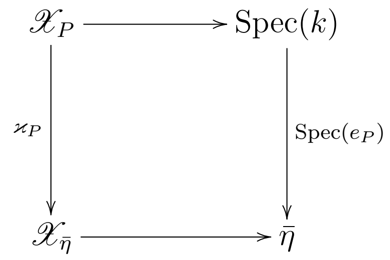
Remark 4.4.2. It is important to mention that the above isomorphism e_P is non-canonical depends on the choice of the transcendental basis B containing the quantities $b_1, \ldots, b_d$.

Let now $f : \mathscr{X} \to S$ be a flat morphism of schemes over k. Extending k_0 if necessary we may well assume that there exists a morphism of schemes $f_0 : \mathscr{X}_0 \to S_0$ over k_0, such that f is the pull-back of f_0 under the field extension from k_0 to k. Let $\eta_0 = \mathrm{Spec}(k_0(S_0))$ be the generic point of the scheme S_0, $\eta = \mathrm{Spec}(k(S))$ the generic point of the scheme S, and $\bar{\eta} = \mathrm{Spec}(\overline{k(S)})$ be the geometric generic point of S. Then we also have the corresponding fibres $\mathscr{X}_{0,\eta_0}$, $\mathscr{X}_\eta$ and $\mathscr{X}_{\bar{\eta}}$.

Pulling back the scheme-theoretic isomorphism $\mathrm{Spec}(e_P)$ onto the fibres of the family f we obtain the Cartesian squares

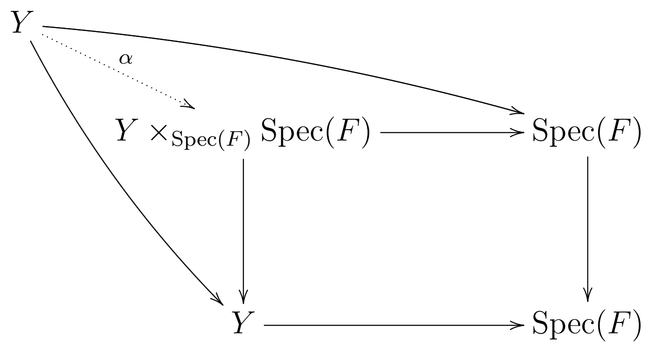


and

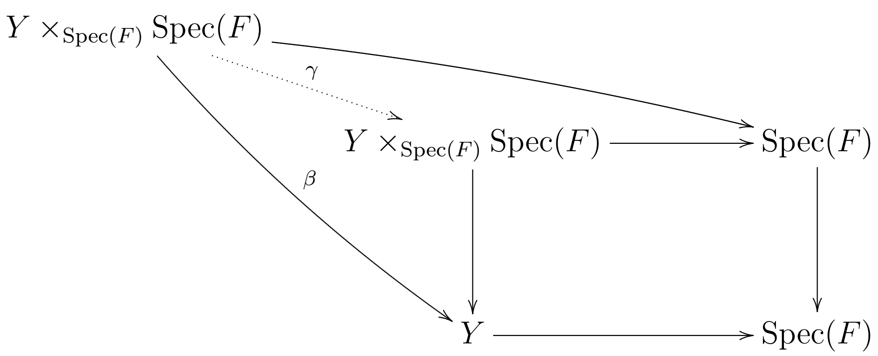


Since $\mathrm{Spec}(e_P)$ is an isomorphism of schemes over η_0, the morphism $\varkappa_P$ is an isomorphism of schemes over $\mathscr{X}_{0,\eta_0}$.

For any field F, a scheme Y over F and an automorphism σ of F let Y_σ be the fibred product of Y and $\mathrm{Spec}(F)$ over $\mathrm{Spec}(F)$, with regard to the automorphism $\mathrm{Spec}(\sigma)$. Let us prove that the scheme Y_σ is scheme-theoretically isomorphic to Y. By the universality of the fibred product there exists a unique morphism α such that the following diagram commutes



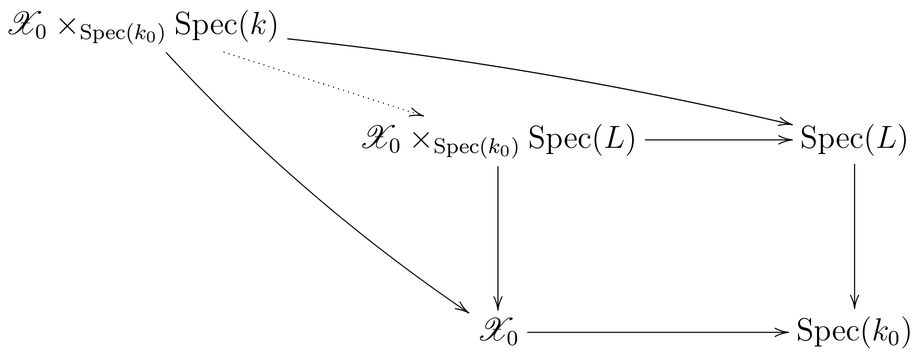
Therefore letting β be the morphism from $Y \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F)$ to Y, we get that the commutativity of the above diagram gives $\beta \circ \alpha = \mathrm{id}$. On the other hand we want to prove that $\alpha \circ \beta = \mathrm{id}$. By the universality we have a morphism γ such that the following diagram commutes



Now $\alpha \circ \beta$ is one such morphism such that $\beta \circ (\alpha \circ \beta) = (\beta \circ \alpha) \circ \beta = \beta$ so the lower triangle commutes in the above diagram, since $\mathrm{Spec}(F)$ is just one point we have that the upper triangle commutes as well. Therefore by the uniqueness of the morphism γ we can say that $\alpha \circ \beta = \mathrm{id}$.

Thus, Y_σ is scheme-theoretically isomorphic to Y, and let $w_\sigma : Y_\sigma \xrightarrow{\sim} Y$ be the corresponding isomorphism of schemes over $\mathrm{Spec}(F^\sigma)$, where F^σ is the subfield of σ-invariant elements in F.

Let L be a field subextension of k/k_0. The projection $\mathscr{X} \to \mathscr{X}_0$ naturally factors through $\mathscr{X}_{0L}$ as follows. By the universality of the fibred product we have the following diagram



where the morphism from $\mathscr{X}_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$ to $\mathscr{X}_0$ is the projection and the diagram is commutative because of the fact that $\mathrm{Spec}(k_0)$ is just one point. This gives us that that the projection $\mathscr{X} \to \mathscr{X}_0$ factors through $\mathscr{X}_{0L} = \mathscr{X}_0 \times_{\mathrm{Spec} k_0} \mathrm{Spec}(L)$. Composing the embedding of the fibre $\mathscr{X}_P$ into the total scheme $\mathscr{X}$ with the morphism $\mathscr{X} \to \mathscr{X}_{0L}$ we can consider $\mathscr{X}_P$ as a scheme over $\mathscr{X}_{0L}$.

If now P' is another closed k-point in U, let $\sigma_{PP'} = e_{P'} \circ e_P^{-1}$ be the automorphism of the field k, and let $\varkappa_{PP'} = \varkappa_{P'}^{-1} \circ \varkappa_P$ be the induced isomorphism of the fibres as schemes over $\mathrm{Spec}(k^{\sigma_{PP'}})$. In these terms, $(\mathscr{X}_P)_{\sigma_{PP'}} = \mathscr{X}_{P'}$, the isomorphism $w_{\sigma_{PP'}} : \mathscr{X}_{P'} \xrightarrow{\sim} \mathscr{X}_P$ is over $\mathscr{X}_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k^{\sigma_{PP'}})$, and $w_{\sigma_{PP'}} = \varkappa_{P'P}$. To see that we just need to use Proposition 4.4.1 and pull-back the scheme-theoretic isomorphisms between points on S to isomorphisms between the corresponding fibres of the morphism $f : \mathscr{X} \to S$.

Remark 4.4.3. The assumption that S is affine is not essential, of course. We can always cover S by open affine subschemes, construct the system of isomorphisms $\varkappa$ in each affine chart and then construct "transition isomorphisms" between very general fibres in a flat family over an arbitrary integral base S.

Let now $f : \mathscr{X} \to S$ and $g : \mathscr{Y} \to S$ be two flat morphisms of schemes over k, and let $r : \mathscr{Y} \to \mathscr{X}$ be a morphism of schemes over the base S, i.e. $f \circ r = g$. Extending k_0 appropriately we may assume that there exist models f_0, g_0 and h_0 over k_0 of the morphisms f, g and h respectively, such that $f_0 \circ r_0 = g_0$. Then, for any closed k-point P in U, the diagram

$$\begin{array}{ccc} \mathscr{Y}_P & \xrightarrow{r_P} & \mathscr{X}_P \\ \downarrow{\scriptstyle \varkappa_P} & & \downarrow{\scriptstyle \varkappa_P} \\ \mathscr{Y}_{\bar{\eta}} & \xrightarrow{r_{\bar{\eta}}} & \mathscr{X}_{\bar{\eta}} \end{array} \tag{4.5}$$

is commutative, where r_P and $r_{\bar{\eta}}$ are the obvious morphisms on fibres induced by the morphism r. Then, of course, the isomorphisms $\varkappa_{PP'}$ commute with the morphisms r_P and $r_{P'}$, for any two closed k-points P and P' in U. Cutting out more Zariski closed subsets from U we may assume that the fibres of the families f and g over the points from U are smooth.

Assume now that k is, moreover, uncountable of characteristic 0, and that Assumptions (A) and (B) are satisfied for the geometric generic fibre $\mathscr{Y}_{\bar{\eta}}$ and the fibre $\mathscr{Y}_P$ for each closed point P in U. To simplify our notation, let $A_{\bar{\eta}}$ be the abelian variety $A^p_{\mathscr{Y}_{\bar{\eta}}}$, let A_P be the abelian variety $A^p_{\mathscr{Y}_P}$, let $\psi_{\bar{\eta}}$ be the universal regular isomorphism $\psi^p_{\mathscr{Y}_{\bar{\eta}}}$ and let $\psi_{\bar{\eta}}$ be the universal regular isomorphism $\psi^p_{\mathscr{Y}_{\bar{\eta}}}$, in terms of Section 4.2.

Lemma 4.4.4. *The scheme-theoretic isomorphism $\varkappa_P : \mathscr{Y}_P \to \mathscr{Y}_{\bar{\eta}}$, constructed in Section 4.4, preserves the algebraic and rational equivalence of algebraic cycles.*

Proof. As we have already explained in Section 4.1, if $\alpha : k \xrightarrow{\sim} k'$ is an isomorphism of fields, the functorial bijections θ from the representation of Chow monoids by Chow schemes commute by means of the isomorphisms of monoids and Hom-sets induced by the isomorphism $\mathrm{Spec}(\alpha)$. To prove that we have to show that the following diagram is commutative

$$\begin{array}{ccc} \mathscr{C}^p(\mathscr{Y}_{\bar{\eta}} \times_{\bar{\eta}} C_{\bar{\eta}}/C_{\bar{\eta}}) & \xrightarrow{\theta_{\mathscr{Y}_{\bar{\eta}}}(C_{\bar{\eta}})} & \mathrm{Hom}_{\bar{\eta}}(C_{\bar{\eta}}, C^p(\mathscr{Y}_{\bar{\eta}}/\bar{\eta})) \\ \downarrow & & \downarrow \\ \mathscr{C}^p(\mathscr{Y}_P \times_{\mathrm{Spec}(k)} C/C) & \xrightarrow{\theta_{\mathscr{Y}_P}(C)} & \mathrm{Hom}_{\mathrm{Spec}(k)}(C, C^p(\mathscr{Y}_P/\mathrm{Spec}(k))) \end{array}$$

In ones turn, to show that it is enough to show that the following diagram is commutative, because there is a natural map of abelian monoids $\mathscr{C}^{eff}(X \times S/S)$ to $\mathrm{Hom}(S, \mathrm{Sym}(X))$, see [34]

$$\begin{array}{ccc} \mathscr{C}^p_d(\mathscr{Y}_{\bar{\eta}} \times_{\bar{\eta}} C_{\bar{\eta}}/C_{\bar{\eta}}) & \xrightarrow{\theta_{\mathscr{Y}_{\bar{\eta}}}(C_{\bar{\eta}})} & \mathrm{Hom}_{\bar{\eta}}(C_{\bar{\eta}}, \mathrm{Sym}^d(\mathscr{Y}_{\bar{\eta}})) \\ \downarrow & & \downarrow \\ \mathscr{C}^p_d(\mathscr{Y}_P \times_{\mathrm{Spec}(k)} C/C) & \xrightarrow{\theta_{\mathscr{Y}_P}(C)} & \mathrm{Hom}_{\mathrm{Spec}(k)}(C, \mathrm{Sym}^d(\mathscr{Y}_P)) \end{array}$$

Let us consider a integral subscheme Z in $\mathscr{Y}_{\bar{\eta}} \times_{\bar{\eta}} C_{\bar{\eta}}$ such that the projection from Z to S is finite and surjective of degree d. Now we consider the morphism

$$s_{Z/C_{\bar{\eta}},d} : C_{\bar{\eta}} \to \mathrm{Sym}^d(Z/C_{\bar{\eta}})$$

and consider the embedding

$$\mathrm{Sym}^d(Z/C_{\bar{\eta}}) \to \mathrm{Sym}^d(\mathscr{Y}_{\bar{\eta}} \times_{\bar{\eta}} C_{\bar{\eta}}/C_{\bar{\eta}}) \cong \mathrm{Sym}^d(\mathscr{Y}_{\bar{\eta}})$$

this gives us $\theta_{\mathscr{Y}_{\bar{\eta}}}(Z)$.

Now consider the following isomorphism

$$k \cong k(\bar{S})$$

that gives us an isomorphism

$$C \to C_{\bar{\eta}} \ ,$$

and

$$\mathscr{Y}_P \to \mathscr{Y}_{\bar{\eta}} \ .$$

Therefore composing we get a morphism

$$C \to C_{\bar{\eta}} \to \mathrm{Sym}^d(Z) \to \mathrm{Sym}^d(\mathscr{Y}_{\bar{\eta}}) \to \mathrm{Sym}^d(\mathscr{Y}_P) \ .$$

On the other hand we have

$$C \to \mathrm{Sym}^d(Z_P) \to \mathrm{Sym}^d(\mathscr{Y}_P) \ ,$$

here Z_P is the pullback of Z, with respect to the morphism

$$\mathrm{Spec}(k) \to \mathrm{Spec}(k(S)) \ .$$

Now to prove the required commutativity we have to prove that the above two compositions are the same. That is we have to prove that the two following diagrams are commutative.

$$\begin{array}{ccc} \mathrm{Sym}^d(Z) & \longrightarrow & \mathrm{Sym}^d(\mathscr{Y}_{\bar{\eta}} \times C_{\bar{\eta}}) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^d(Z_P) & \longrightarrow & \mathrm{Sym}^d(\mathscr{Y}_P \times C_P) \end{array}$$

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Sym}^d(Z_P) \\ \downarrow & & \uparrow \\ C_{\bar{\eta}} & \longrightarrow & \mathrm{Sym}^d(Z) \end{array}$$

The commutativity of the first diagram follows from the commutativity of the diagram.

$$\begin{array}{ccc} Z & \longrightarrow & \mathscr{Y}_{\bar{\eta}} \times C_{\bar{\eta}} \\ \downarrow & & \downarrow \\ Z_P & \longrightarrow & \mathscr{Y}_P \times C_P \end{array}$$

So we only have to show the commutativity of the second diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Sym}^d(Z_P) \\ \downarrow & & \uparrow \\ C_{\bar{\eta}} & \longrightarrow & \mathrm{Sym}^d(Z) \end{array}$$

to do that we reduce everything to the affine case, write $C = \mathrm{Spec}(B)$ and $Z_P = \mathrm{Spec}(A)$ then we are reduced to show that the following diagram in the category of commutative rings is commutative.

$$\begin{array}{ccc} B \otimes_k k(\bar{S}) & \longleftarrow & \mathrm{Sym}^d(A \otimes_k k(\bar{S})) \\ \downarrow & & \uparrow \\ B & \longleftarrow & \mathrm{Sym}^d(A) \end{array}$$

Now consider the element $a_1 \otimes \cdots \otimes a_d$ in $\mathrm{Sym}^d(A)$. The right vertical homomorphism sends it to

$$(a_1 \otimes 1) \otimes \cdots \otimes (a_d \otimes 1)$$

and the top horizontal morphism sends it to

$$(a_1 \otimes 1) \wedge \cdots \wedge (a_d \otimes 1)$$

now we have to prove that this is equal to

$$(a_1 \wedge \cdots \wedge a_d) \otimes 1 \ ,$$

then we will be done with the commutativity of the above diagram. To show that we show,

$$(a_1 \otimes 1) \otimes \cdots \otimes (a_d \otimes 1) - (a_1 \otimes \cdots \otimes a_d) \otimes 1 = 0$$

but that follows from writing $1 = 1 \otimes \cdots \otimes 1$ for d-many times. Therefore we have the commutativity of the required diagram.

In particular, if $k' = \overline{k(S)}$ and $\alpha = e_P^{-1}$, the bijections $\theta_{\mathscr{Y}_{\bar\eta}}$ over $\overline{k(S)}$ commute with the bijections $\theta_{\mathscr{Y}_P}$ over k. The commutativity for the sections of the corresponding pre-sheaves on an algebraic curve C over k and its pull-back C' over k' gives the first assertion of the lemma.

If $C = \mathbb{P}^1$, we get the second one. □

By Lemma 4.4.4, the isomorphism $\varkappa_P$ induces the push-forward isomorphism of abelian groups $\varkappa_{P*} : A^p(\mathscr{Y}_P) \to A^p(\mathscr{Y}_{\bar\eta})$. Let $\kappa_P : A_P \to A_{\bar\eta}$ be the composition $\psi^p_{\bar\eta} \circ \varkappa_{P*} \circ (\psi^p_P)^{-1}$. We will be saying that Assumption (A) is satisfied *in a family*, for the fibres $\mathscr{Y}_P$ at the closed points P in U, if there exists an abelian scheme $\mathscr{A}$ over a Zariski open subscheme W' in some finite extension S' of S, such that the fibre $\mathscr{A}_P$ of this abelian scheme at $P' \in W'$ over $P \in U$ is A_P, the fibre $\mathscr{A}_{\bar\eta}$ at the geometric generic point $\bar\eta$ of the family $\mathscr{A} \to W'$ is $A_{\bar\eta}$, and the corresponding scheme-theoretic isomorphism

$$\varkappa_{P'*} : A_P \stackrel{\sim}{\to} A_{\bar\eta}$$

coincides with the above constructed isomorphism κ_P, for each closed point P in U. This gives, in particular, that κ_P is a regular morphism of schemes, for each P in U.

Notice that the satisfaction of Assumption (A) in a family can be always achieved provided Assumption (A) holds for the geometric generic fibre $\mathscr{Y}_{\bar\eta}$. Indeed, as $\psi^p_Y : A^p(Y) \to A^p_Y$ is an isomorphism, for $Y = \mathscr{Y}_{\bar\eta}$, the group $A^p(Y)$ is weakly representable. As we mentioned above, in such a situation we can choose a smooth projective curve Γ over $\bar\eta$ and a cycle Z on $\Gamma \times Y$ inducing a surjective homomorphism from $A^1(\Gamma)$ onto $A^p(Y)$ over $\bar\eta$. Then ψ^p_Y can be fully described geometrically in terms of the Jacobian J_Γ of the curve curve Γ and the cycle Z. Spreading out Γ, J_Γ and Z over an open subscheme W' in a suitable finite extension S' of S, and then specializing the relative construction to a closed point P of the c-open set U, we get weak representability of $A^p(\mathscr{Y}_P)$ provided by the specializations of the spread of the cycle Z to P.

Let us give more details on it. By Assumption (A) we get that there exists a regular isomorphism

$$\psi^p_Y : A^p(Y) \to A^p_Y \ ,$$

where A^p_Y is an abelian variety. By definition of weak representability we get that there exists a smooth projective curve Γ over $\bar\eta$ and a cycle Z supported on $\Gamma \times Y$ inducing a surjective homomorphism

$$Z_* : A^1(\Gamma) \to A^p(\mathscr{Y}_{\bar\eta}) \ .$$

Since $A^1(\Gamma)$ is isomorphic to the Jacobian J_Γ and ψ^p_Y is regular we get that the $\psi^p_Y \circ Z_*$ is regular.

Now let L be the finitely generated field over $k(S)$, over which Γ, J_Γ, Z are all defined. Then consider the spreads of

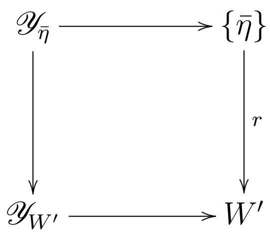
$$\Gamma\,, \quad J_\Gamma \quad \text{and} \quad Z$$

over some open subscheme W' of S', where

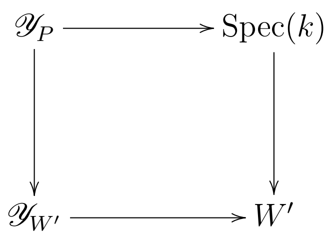
$$k(S') = L\,.$$

Let us denote these spreads by $\mathscr{G}$, $\mathscr{J}$ and $\mathscr{Z}$ respectively.

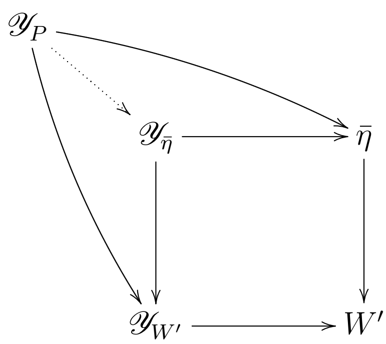
Now consider the Cartesian squares



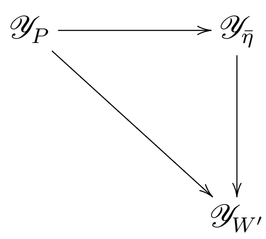
and the Cartesian square



By the universality of fibred products we get following the commutative diagram



That gives us the commutative triangle



Now we have the following commutative diagram at the level of Chow groups.

$$\begin{array}{ccc}
A^1(\Gamma_{\bar{\eta}}) & \xrightarrow{Z_{\bar{\eta}*}} & A^p(\mathscr{Y}_{\bar{\eta}}) \\
\uparrow & & \uparrow {\scriptstyle f^*_{\bar{\eta}}} \\
A^1(\mathscr{G}) & \xrightarrow{\mathscr{Z}_*} & A^p(\mathscr{Y}_{W'}) \\
\downarrow & & {\scriptstyle f^*_P} \downarrow \\
A^1(\mathscr{G}_P) & \xrightarrow{\mathscr{Z}_{P*}} & A^p(\mathscr{Y}_P)
\end{array}$$

It is easy to show that the two squares present in the above rectangle are commutative and that the group $A^p(\mathscr{Y}_{W'})$ is generated by the image of $\mathscr{Z}_*$ and the kernel of f^*. Now we have to prove that the homomorphism $\mathscr{Z}_{P*}$ is surjective from $A^1(\mathscr{G}_P)$ to $A^p(\mathscr{Y}_P)$. Since the triangle

$$\begin{array}{ccc}
A^p(\mathscr{Y}_{W'}) & \longrightarrow & A^p(\mathscr{Y}_{\bar{\eta}}) \\
& \searrow & \downarrow \\
& & A^p(\mathscr{Y}_P)
\end{array}$$

is commutative and $A^p(\mathscr{Y}_{W'}) \to A^p(\mathscr{Y}_{\bar{\eta}})$ is surjective we have that

$$A^p(\mathscr{Y}_{W'}) \to A^p(\mathscr{Y}_P)$$

is surjective. So take b in $A^p(\mathscr{Y}_P)$, then there exists a in $A^p(\mathscr{Y}_{W'})$ such that

$$f^*_P(a) = b$$

Now we can write a as $\mathscr{Z}_*(c) + d$ where d belongs to the kernel of $f^*_{\bar{\eta}}$. Let e be the image of c under the homomorphism

$$A^1(\mathscr{G}) \to A^1(\mathscr{G}_P)\ .$$

Now since

$$\begin{array}{ccc}
A^p(\mathscr{Y}_{W'}) & \longrightarrow & A^p(\mathscr{Y}_{\bar{\eta}}) \\
& \searrow & \downarrow \\
& & A^p(\mathscr{Y}_P)
\end{array}$$

is commutative we get that

$$f_P^*(d) = 0$$

and then by the commutativity of the lower part of the above rectangle it follows that

$$\mathscr{Z}_{P*}(e) = b \ .$$

So the homomorphism $\mathscr{Z}_{P*}$ is surjective. Considering the spread of the abelian variety $A_{\bar{\eta}}$, we get an abelian scheme $\mathscr{A}$ over W' and since $A_{\bar{\eta}} \cong A_P$ for a closed point P in U and also

$$\mathscr{A}_{\bar{\eta}} \cong \mathscr{A}_P$$

which gives us that

$$\mathscr{A}_P \cong A_P \ .$$

This gives us Assumption (A) satisfied in a family.

Thus, in order to have Assumption (A) in a family, all we need is to assume that it holds for the geometric generic fibre $\mathscr{Y}_{\bar{\xi}}$. It is essential that we only take care about the closed points of the c-open set U, for which we have the scheme-theoretical isomorphisms $\varkappa_P$. If we choose a closed point P beyond U, then weak representability cannot be guaranteed.

Similarly, if we assume that Assumption (B) is satisfied for the geometric generic fibre $\mathscr{Y}_{\bar{\eta}}$, then it will be also satisfied for each closed point P in U. This can be proven by means of the argument used in the proof of Lemma 4.4.4.

Indeed, since the condition B is satisfied for the geometric generic fiber we have that, the quotient group of algebraic cycles of codimension p on $\mathscr{Y}_{\bar{\eta}}$ modulo the algebraically trivial algebraic cycles is $\mathbb{Z}$. Consider P in U and consider an algebraic cycle on $\mathscr{Y}_P$ such that the degree of the cycle is zero. Then we have to prove that the cycle is algebraically trivial. Since $\mathscr{Y}_P$ is isomorphic to $\mathscr{Y}_{\bar{\eta}}$ as schemes over $\bar{\eta}$ we have that the group of algebraic cycles of codimension p on $\mathscr{Y}_P$ is isomorphic to group of algebraic cycles of codimension p on $\mathscr{Y}_{\bar{\eta}}$, and the same for the algebraically trivial cycles of codimension p on $\mathscr{Y}_P$ and $\mathscr{Y}_{\bar{\eta}}$. Therefore the degree zero algebraic cycles on $\mathscr{Y}_P$ correspond to the degree zero algebraic cycles on $\mathscr{Y}_{\bar{\eta}}$.

But since the group of algebraic cycles modulo the group of algebraically trivial algebraic cycles is $\mathbb{Z}$ we have that degree zero algebraic cycles on $\mathscr{Y}_{\bar{\eta}}$ are algebraically trivial. But by Lemma 4.4.4 the morphism $\varkappa_P$ preserves rational and algebraic equivalence.

Therefore we get that the degree zero algebraic cycles on $\mathscr{Y}_P$ are algebraically trivial. This follows from the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathscr{C}^p(\mathscr{Y}_{\bar\eta} \times_{\bar\eta} C_{\bar\eta}/C_{\bar\eta}) & \xrightarrow{\theta_{\mathscr{Y}_{\bar\eta}}(C_{\bar\eta})} & \mathrm{Hom}_{\bar\eta}(C_{\bar\eta}, C^p(\mathscr{Y}_{\bar\eta}/\bar\eta)) \\
\downarrow & & \downarrow \\
\mathscr{C}^p(\mathscr{Y}_P \times_{\mathrm{Spec}(k)} C/C) & \xrightarrow{\theta_{\mathscr{Y}_P}(C)} & \mathrm{Hom}_{\mathrm{Spec}(k)}(C, C^p(\mathscr{Y}_P/\mathrm{Spec}(k)))
\end{array}$$

Next, the commutative diagram (4.5) gives the commutative diagram

$$\begin{array}{ccc}
A^p(\mathscr{Y}_P) & \xrightarrow{r_{P*}} & A^{p+e}(\mathscr{X}_P) \\
\varkappa_{P*}\downarrow & & \downarrow\varkappa_{P*} \\
A^p(\mathscr{Y}_{\bar\eta}) & \xrightarrow{r_{\bar\eta*}} & A^{p+e}(\mathscr{X}_{\bar\eta})
\end{array} \qquad (4.6)$$

where

$$e = \dim(\mathscr{X}_{\bar\eta}) - \dim(\mathscr{Y}_{\bar\eta})\ .$$

To shorten notation further, let

$$A_{P,0}$$

be the abelian subvariety in A_P and

$$A_{\bar\eta,0}$$

the abelian subvariety in $A_{\bar\eta}$, such that, by Proposition 4.2.4, the image

$$K_P$$

of the kernel of the push-forward homomorphism

$$r_{P*} : A^p(\mathscr{Y}_P) \to A^{p+e}(\mathscr{X}_P)$$

under the isomorphism

$$\psi_P : A^p(\mathscr{Y}_P) \stackrel{\sim}{\to} A_P$$

is the union of a countable collection of shifts of $A_{P,0}$ inside A_P.

Similarly, the image

$$K_{\bar\eta}$$

of the kernel of the homomorphism $r_{\bar\eta_*}$ under the isomorphism $\psi_{\bar\eta}$ is the union of a countable collection of shifts of the abelian subvariety $A_{\bar\eta,0}$ in $A_{\bar\eta}$.

Proposition 4.4.5. *Suppose that Assumption (A) is satisfied for the fibre $\mathscr{Y}_{\bar\eta}$. Then, for any closed point P in U, $\kappa_P(A_{P,0}) = A_{\bar\eta,0}$.*

Proof. Let Ξ_P be the countable subset in A_P and $\Xi_{\bar{\eta}}$ the countable subset in $A_{\bar{\eta}}$, such that

$$K_P = \cup_{x\in\Xi_P}(s + A_{P,0})$$

and

$$K_{\bar{\eta}} = \cup_{x\in\Xi_{\bar{\eta}}}(x + A_{\bar{\eta},0})$$

in A_P and $A_{\bar{\eta}}$ respectively (see Proposition 4.2.4). Then

$$\kappa_P(K_P) = \kappa_P(\cup_{x\in\Xi_P}(x + A_{P,0})) = \cup_{x\in\Xi_P}(\kappa_P(x) + \kappa_P(A_{P,0})) \ .$$

The definition of κ_P and the commutative diagram (4.6) give that

$$\kappa_P(K_P) = K_{\bar{\eta}} \ .$$

Therefore, the union

$$\cup_{x\in\Xi_P}(\kappa_P(x) + \kappa_P(A_{P,0}))$$

coincides with the union

$$\cup_{x\in\Xi_{\bar{\eta}}}(x + A_{\bar{\eta},0})$$

inside the abelian variety $A_{\bar{\eta}}$.

Since Assumption (A) is satisfied for the fibre at $\bar{\eta}$, it is satisfied in a family. Therefore, the homomorphisms κ_P are regular morphisms of schemes, whence $\kappa_P(A_{P,0})$ is a Zariski closed subset in $A_{\bar{\eta}}$.

Since $\kappa_P(A_{P,0})$ is a subgroup in $A_{\bar{\eta}}$, it is an abelian subvariety in $A_{\bar{\eta}}$. Lemma 4.2.2 and Lemma 4.2.3 finish the proof. □

Remark 4.4.6. Assume that $p = 1$ or 2, and $e = 1$. Let $A_{\bar{\eta},1}$ be the abelian subvariety in $A_{\bar{\eta}}$ constructed as in Section 4.3. For each closed point P in U one can define $A_{P,1}$ to be the pre-image of $A_{\bar{\eta},1}$ under the isomorphism κ_P. Proposition 4.4.5 gives that $A_{P,0}$ is an abelian subvariety in $A_{P,1}$. If

$$H^{2p+1}_{\acute{e}t}(\mathscr{X}_{\bar{\eta}}, \mathbb{Q}_l) = 0$$

then $A_{\bar{\eta},1} = A_{\bar{\eta}}$ and $A_{P,1} = A_P$ for each closed point P in U.

4.5 Cycles on hyperplane sections via étale monodromy

Let k be an uncountable algebraically closed field of characteristic zero. Let $d = 2p$ and let $\mathscr{X}$ be a nonsingular d-dimensional projective variety over the ground field k. Fix a closed embedding $\mathscr{X} \subset \mathbb{P}^m$, such that $\mathscr{X}$ is not contained in a smaller linear subspace in $\mathbb{P}^m$. For any closed point t in the dual projective space $\mathbb{P}^{m\vee}$, let H_t be the corresponding hyperplane in $\mathbb{P}^m$ and let $\mathscr{X}_t$ be the intersection of $\mathscr{X}$ with H_t. Let

$$\mathscr{H} = \{(P, H) \in \mathbb{P}^m \times \mathbb{P}^{m\vee} \mid P \in H\}$$

be the universal hyperplane, and let p_1 and p_2 be the projections of $\mathscr{H}$ on $\mathbb{P}^m$ and $\mathbb{P}^{m\vee}$ respectively.

Let

$$\mathscr{Y} \to \mathscr{X}$$

be the pull-back of p_1 with respect to the embedding $\mathscr{X} \subset \mathbb{P}^m$, and let

$$f : \mathscr{Y} \to \mathbb{P}^{m\vee}$$

be the composition of the closed embedding of $\mathscr{Y}$ into $\mathscr{H}$ with the projection p_2.

For any morphism of schemes

$$D \to \mathbb{P}^{m\vee}$$

let

$$\mathscr{H}_D \to D$$

be the pull-back of p_2 with respect to the morphism $D \to \mathbb{P}^{m\vee}$, let $\mathscr{Y}_D$ be the fibred product of $\mathscr{Y}$ and $\mathscr{H}_D$ over the universal hyperplane $\mathscr{H}$, and let

$$f_D : \mathscr{Y}_D \to D$$

be the induced projection, i.e. the composition of the closed embedding of $\mathscr{Y}_D$ into $\mathscr{H}_D$ and the morphism $\mathscr{H}_D \to D$. Let also $\mathscr{X}_D \to D$ be the pull-back of trivial family

$$\mathscr{X}_{\mathbb{P}^{m\vee}} = \mathscr{X} \times_{\mathrm{Spec}(k)} \mathbb{P}^{m\vee} \to \mathbb{P}^{m\vee}$$

with respect to the morphism $D \to \mathbb{P}^{m\vee}$.

Now, choose D to be a projective line in $\mathbb{P}^{m\vee}$, such that the morphism f_D is a Lefschetz pencil of the variety $\mathscr{X}$. Let $k(D)$ be the function field of D,

$$\eta = \mathrm{Spec}(k(D))$$

be the generic point of D, $\overline{k(D)}$ be the algebraic closure of $k(D)$ and

$$\bar{\eta} = \mathrm{Spec}(\overline{k(D)})$$

be the geometric generic point of D.

Let Y be the geometric generic fibre $\mathscr{Y}_{\bar{\eta}}$ of the morphism $f_D : \mathscr{Y}_D \to D$, and let X be the geometric generic fibre of the morphism $\mathscr{X}_D \to D$.

The universal morphism $\mathscr{Y}_D \to \mathscr{X}_D$ is over D, so that it induces the morphism

$$r : Y = \mathscr{Y}_{\bar{\eta}} \to X = \mathscr{X}_{\bar{\eta}}$$

over the geometric generic point $\bar{\eta}$. The morphism r is a closed imbedding of the scheme Y into the scheme X.

Assume that the variety $Y = \mathscr{Y}_{\bar{\eta}}$ satisfies Assumptions (A) and (B), being considered over $\overline{k(D)}$. That is, the universal homomorphism $\psi^p_Y = \psi^p_{\mathscr{Y}_{\bar{\eta}}}$ exists and is an isomorphism of abelian groups, and the quotient-group of algebraic cycles of codimension p on Y modulo algebraically trivial algebraic cycles is $\mathbb{Z}$.

Let

$$r_* : A^p(Y) \to A^{p+1}(X)$$

be the push-forward homomorphism induced by the proper morphism r, and let K^p_r be the image of the kernel of r_* under the isomorphism ψ^p_Y.

Let also

$$A_{\bar{\eta}}$$

be the abelian variety

$$A^p_Y = A^p_{\mathscr{Y}_{\bar{\eta}}}$$

and let

$$A_{\bar{\eta},0} \quad \text{and} \quad A_{\bar{\eta},1}$$

be the abelian subvarieties in $A_{\bar{\eta}}$, as in the previous section.

The image of the composition of the homomorphism

$$\zeta_{\mathbb{Q}_l} : H^1_{\acute{e}t}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p)) \to H^1_{\acute{e}t}(A_{\bar{\eta}}, \mathbb{Q}_l(1-p))$$

with the homomorphism

$$w_* : H^1_{\acute{e}t}(A_{\bar{\eta}}, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l)$$

is contained in the kernel of the Gysin homomorphism

$$r_* : H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l) \to H^{2p+1}_{\acute{e}t}(Y, \mathbb{Q}_l)$$

due to Proposition 4.3.1. This is why

$$A_{\bar{\eta},0} \subset A_{\bar{\eta},1} \subset A_{\bar{\eta}} \ .$$

Let now L be the minimal subextension of $k(D)$ in $\overline{k(D)}$, such that the abelian varieties

$$A_{\bar{\eta},0}\ , \quad A_{\bar{\eta},1} \quad \text{and} \quad A_{\bar{\eta}}$$

are all defined over L. Then L is finitely generated and algebraic of finite degree n over $k(D)$. Let D' be an algebraic curve, such that

$$L = k(D')$$

and the embedding of k into $k(D)$ is induced by a generically of degree n morphism from D' onto D.

Since the closed embedding of

$$A_{\bar{\eta},0} \subset A_{\bar{\eta},1}$$

and

$$A_{\bar{\eta},1} \subset A_{\bar{\eta}}$$

are now defined over L, there exist a Zariski open subset U' in D', spreads

$$\mathscr{A}_{\bar{\eta},0}\ ,\quad \mathscr{A}_{\bar{\eta},1} \quad \text{and} \quad \mathscr{A}_{\bar{\eta}}$$

of $A_{\bar{\eta},0}$, $A_{\bar{\eta},1}$ and $A_{\bar{\eta}}$ respectively over U', and morphisms

$$\mathscr{A}_{\bar{\eta},0} \to \mathscr{A}_{\bar{\eta},1}$$

and

$$\mathscr{A}_{\bar{\eta},1} \to \mathscr{A}_{\bar{\eta}}$$

over U', such that, when passing to the fibres at the geometric generic point $\bar{\eta}$, we obtain the closed embeddings of $A_{\bar{\eta},0}$ into $A_{\bar{\eta},1}$ and $A_{\bar{\eta},1}$ into $A_{\bar{\eta}}$ over $\overline{k(D)}$.

Let α be the morphism from $\mathscr{A}$ onto U', and let α_0 and α_1 be the morphism from $\mathscr{A}_0$ and, respectively, $\mathscr{A}_1$ onto U'. Since $\mathscr{A}$ is a spread of $A_{\bar{\eta}}$ over U' and $A_{\bar{\eta}}$ is a projective variety over $L = k(D')$, the morphism α is locally projective and, therefore, proper. Similarly, the morphisms α_0 and α_1 are proper. Cutting more points from D' we may assume that the morphisms α, α_0 and α_1 are all smooth over U'.

Let

$$\eta' = \mathrm{Spec}(k(D'))$$

be the generic point of D', let

$$\bar{\eta}' = \bar{\eta}$$

be the geometric generic point of D', let

$$\pi_1(U', \bar{\eta})$$

be the étale fundamental group of D' pointed at $\bar{\eta}$, and let

$$\pi_1^{\mathrm{tame}}(U', \bar{\eta})$$

be the corresponding tame fundamental group. For any scheme S and non-negative integer n let $(\mathbb{Z}/l^n)_S$ be the constant sheaf on S associated to the group $\mathbb{Z}/l^n$.

Since the morphisms α_0, α_1 and α are smooth and proper, the higher direct images

$$R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathscr{A}_0}\ ,\quad R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}_1} \quad \text{and} \quad R^1\alpha_*(\mathbb{Z}/l^n)_{\mathscr{A}}$$

are locally constant by Theorem 8.9, Ch. I in [12]. Then the fibres

$$(R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathscr{A}_0})_{\bar\eta}\ ,\ \ (R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}_1})_{\bar\eta}\quad\text{and}\quad (R^1\alpha_{*}(\mathbb{Z}/l^n)_{\mathscr{A}})_{\bar\eta}$$

of these sheaves at the geometric generic point $\bar\eta$ are finite continuous $\pi_1(U',\bar\eta)$-modules, see Proposition A I.7 in loc. cit. The proper base change (see, for example, Theorem 6.1$'$ on page 62 in loc. cit.) gives that

$$(R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathscr{A}_0})_{\bar\eta}\quad\text{is}\quad H^1_{\acute{e}t}(\mathscr{A}_{0\bar\eta},\mathbb{Z}/l^n)\ ,$$

$$(R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}_1})_{\bar\eta}\quad\text{is}\quad H^1_{\acute{e}t}(\mathscr{A}_{1\bar\eta},\mathbb{Z}/l^n)$$

and

$$(R^1\alpha_{*}(\mathbb{Z}/l^n)_{\mathscr{A}})_{\bar\eta}\quad\text{is}\quad H^1_{\acute{e}t}(\mathscr{A}_{\bar\eta},\mathbb{Z}/l^n)\ .$$

Then we obtain that $\pi_1(U',\bar\eta)$ acts continuously on

$$H^1_{\acute{e}t}(\mathscr{A}_{0\bar\eta},\mathbb{Z}/l^n)\ ,\ \ H^1_{\acute{e}t}(\mathscr{A}_{1\bar\eta},\mathbb{Z}/l^n)\quad\text{and}\quad H^1_{\acute{e}t}(\mathscr{A}_{\bar\eta},\mathbb{Z}/l^n)\ .$$

Passing to limits on n and then tensoring with $\mathbb{Q}_l$ we then obtain that the étale fundamental group $\pi_1(U',\bar\eta)$ acts continuously on

$$H^1_{\acute{e}t}(\mathscr{A}_{0\bar\eta},\mathbb{Q}_l)=H^1_{\acute{e}t}(A_{\bar\eta,0},\mathbb{Q}_l)\ ,$$

$$H^1_{\acute{e}t}(\mathscr{A}_{1\bar\eta},\mathbb{Q}_l)=H^1_{\acute{e}t}(A_{\bar\eta,1},\mathbb{Q}_l)$$

and

$$H^1_{\acute{e}t}(\mathscr{A}_{\bar\eta},\mathbb{Q}_l)=H^1_{\acute{e}t}(A_{\bar\eta},\mathbb{Q}_l)\ .$$

The homomorphism $\zeta_{\mathbb{Q}_l}$ from $H^1_{\acute{e}t}(A_{\bar\eta,0},\mathbb{Q}_l)$ to $H^1_{\acute{e}t}(A_{\bar\eta},\mathbb{Q}_l)$ is the composition of the obvious homomorphisms $\zeta'_{\mathbb{Q}_l}$ from $H^1_{\acute{e}t}(A_{\bar\eta,0},\mathbb{Q}_l)$ to $H^1_{\acute{e}t}(A_{\bar\eta,1},\mathbb{Q}_l)$ and $\zeta''_{\mathbb{Q}_l}$ from $H^1_{\acute{e}t}(A_{\bar\eta,1},\mathbb{Q}_l)$ to $H^1_{\acute{e}t}(A_{\bar\eta},\mathbb{Q}_l)$. The action of $\pi_1(U',\bar\eta)$ naturally commutes with both $\zeta'_{\mathbb{Q}_l}$ and $\zeta''_{\mathbb{Q}_l}$.

Without loss of generality, we may assume that U' is the pre-image of a Zariski open subset U in D and all the fibres of the Lefschetz pencil $f_D:\mathscr{Y}_D\to D$ over the closed points of U are smooth.

Let

$$f_{D'}:\mathscr{Y}_{D'}\to D'$$

be the pull-back of the Lefschetz pencil f_D with respect to the morphism $D'\to D$, and let

$$f_{U'}:\mathscr{Y}_{U'}\to U'$$

be the pull-back of $f_{D'}$ with respect to the open embedding of U' to D'.

Applying the same arguments to the morphism $f_{U'}$, we obtain the continuous action of the étale fundamental group $\pi_1(U',\bar\eta)$ on the group $H^{2p-1}_{\acute{e}t}(\mathscr{Y}_{\bar\eta},\mathbb{Q}_l)$, and it is well known that this action is tame, in the sense that it factorizes through the surjective homomorphism from $\pi_1(U',\bar\eta)$ onto $\pi_1^{\mathrm{tame}}(U',\bar\eta)$.

For each closed point s in the complement $D \smallsetminus U$ let δ_s be the unique up-to conjugation vanishing cycle in $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$, corresponding to the point s in the standard sense (see Theorem 7.1 on page 247 in [12]), and let E be the $\mathbb{Q}_l$-vector subspace in $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$ generated by all the elements

$$\delta_s \ , \quad s \in D \smallsetminus U \ .$$

In other words, E is the space of vanishing cycles in $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$. One can show that E coincides with the kernel of the Gysin homomorphism r_* from $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$ to $H^{2p+1}_{ét}(\mathscr{X}_{\bar\eta}, \mathbb{Q}_l)$, see Section 4.3 in [10].

In what follows we will be using the étale l-adic Picard-Lefschetz formula for the monodromy action. For each $s \in D \smallsetminus U$ let $\pi_{1,s}$ be the so-called tame fundamental group at s, which is a subgroup in $\pi_1^{\mathrm{tame}}(U, \bar\eta)$ uniquely determined by the point s up to conjugation in $\pi_1^{\mathrm{tame}}(U, \bar\eta)$.

In terms of [12], $\pi_{1,s}$ is the image of the homomorphism

$$\gamma_s : \hat{\mathbb{Z}}(1) \to \pi_1^{\mathrm{tame}}(U, \bar\eta) \ ,$$

where $\hat{\mathbb{Z}}(1)$ is the limit of all groups μ_n, and μ_n is the group of n-th roots of unity in the algebraically closed field $\overline{k(U)}$ whose exponential characteristic is 1.

The tame fundamental group $\pi_1^{\mathrm{tame}}(U, \bar\eta)$ is generated by the subgroups $\pi_{1,s}$. If u is an element in $\hat{\mathbb{Z}}(1)$, let $\bar u$ be the image of u in $\mathbb{Z}_l(1)$. If now v is an element in the $\mathbb{Q}_l$-vector space $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$ the Picard-Lefschetz formula says

$$\gamma_s(u)x = x \pm \bar u \langle x, \delta_s \rangle \delta_s \ . \tag{4.7}$$

Since present we will be assuming that either $p = 1$ or 2.

Proposition 4.5.1. *Under the above assumptions, either $A_{\bar\eta,0} = 0$ or $A_{\bar\eta,0} = A_{\bar\eta,1}$.*

Proof. By Proposition 4.3.1 and the fact that the space E of vanishing cycles coincides with the kernel of the Gysin homomorphism r_* from $H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$ to $H^{2p+1}_{ét}(\mathscr{X}_{\bar\eta}, \mathbb{Q}_l)$, we see that the image of the composition

$$H^1_{ét}(A_{\bar\eta,0}, \mathbb{Q}_l(1-p)) \stackrel{\zeta_{\mathbb{Q}_l}}{\to} H^1_{ét}(A_{\bar\eta}, \mathbb{Q}_l(1-p)) \stackrel{w_*}{\to} H^{2p-1}_{ét}(\mathscr{Y}_{\bar\eta}, \mathbb{Q}_l)$$

is contained in E. The homomorphism $\zeta_{\mathbb{Q}_l}$ is injective and compatible with the action of $\pi_1(U', \bar\eta)$. Since $p \leq 2$, the homomorphism w_* is bijective, see Remark 4.3.2. Then

$$E \simeq H^1_{ét}(A_{\bar\eta,1}, \mathbb{Q}_l(1-p))$$

via $\zeta''_{\mathbb{Q}_l}$ and w_*.

Since the variety $Y = \mathscr{Y}_{\bar\eta}$ satisfies Assumption (A), there exists a nonsingular projective curve Γ and an algebraic cycle Z on $\Gamma \times Y$ over $\bar\eta$, such that the cycle class z of Z induces the surjective homomorphism z_* from $A^1(\Gamma)$ to $A^p(\mathscr{Y}_{\bar{\eta}})$, whose kernel is G. The homomorphism

$$w_* : H^1_{\acute{e}t}(A_{\bar{\eta}}, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\acute{e}t}(\mathscr{Y}_{\bar{\eta}}, \mathbb{Q}_l)$$

is then induced by the composition of the embedding of the curve Γ into its Jacobian J_Γ over $\bar{\eta}$, the quotient map from J_Γ onto the abelian variety $A = J_\Gamma$, also over $\bar{\eta}$, and the homomorphism induced by the correspondence Z (see Section 4.2).

Spreading out the morphisms $\Gamma \to J_\Gamma$ and $J_\Gamma \to A$, as well as the cycle X, over a certain Zariski open subset in D', we can achieve that the homomorphism w_* is compatible with the action of the fundamental group $\pi_1(U', \bar{\eta})$.

Indeed, let Z be an algebraic cycle on $\Gamma \times Y$ and

$$Z_* : J_\Gamma \to A^p(Y)$$

is onto, now we have the universal regular homomorphism

$$\psi^p_Y : A^p(Y) \to A$$

we get that $\psi^p_Y \circ Z_*$ is a regular map, we denote it by Z_* again.

We have an embedding $\Gamma \to J_\Gamma$ and the morphism $Z_* : J_\Gamma \to A$ induced by the cycle Z on $\Gamma \times Y$.

Let

$$\Gamma\ ,\quad J_\Gamma\ ,\quad A \quad \text{and} \quad Z$$

be all defined over some finite extension L of $k(D)$. Choose a curve D' such that

$$k(D') = L$$

and

$$D' \to D$$

is finite.

Let U' be a Zariski open set in D' and we spread Γ, J_Γ, A and Z over U'. Let

$$\mathscr{C}\ ,\quad \mathscr{J}\ ,\quad \mathscr{A} \quad \text{and} \quad \mathscr{Z}$$

denote the corresponding spreads. Then we have the following morphisms

$$\alpha : \mathscr{C} \times \mathscr{Y} \to \mathscr{C}$$

$$\beta : \mathscr{C} \to \mathscr{J}$$

$$\gamma : \mathscr{J} \to \mathscr{A}\ .$$

Now consider the constant sheaves

$$(\mathbb{Z}/l^n)_{\mathscr{A}}\ ,\quad (\mathbb{Z}/l^n)_{\mathscr{J}} \quad \text{and} \quad (\mathbb{Z}/l^n)_{\mathscr{C}}$$

on $\mathscr{A}$, $\mathscr{J}$ and $\mathscr{C}$ respectively. Let also

$$p_1\ ,\quad p_2\ ,\quad p_3 \quad \text{and} \quad p_4$$

denote the morphisms from $\mathscr{A}$, $\mathscr{J}$, $\mathscr{C}$ and $\mathscr{C}\times\mathscr{Y}$ to U' respectively.

Then we have

$$R^1(\gamma^*) : R^1p_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}} \to R^1p_{2*}(\mathbb{Z}/l^n)_{\mathscr{J}}$$

$$R^1(\beta^*) : R^1p_{2*}(\mathbb{Z}/l^n)_{\mathscr{J}} \to R^1p_{3*}(\mathbb{Z}/l^n)_{\mathscr{C}}$$

$$R^1(\alpha^*) : R^1p_{3*}(\mathbb{Z}/l^n)_{\mathscr{C}} \to R^1p_{4*}(\mathbb{Z}/l^n)_{\mathscr{C}\times\mathscr{Y}}\ .$$

Composing we get

$$R^1((\gamma\circ\beta\circ\alpha)^*) : R^1p_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}} \to R^1p_{4*}(\mathbb{Z}/l^n)_{\mathscr{C}\times\mathscr{Y}}$$

Passing to the stalks of the above higher direct image sheaves we get that

$$R^1((\gamma\circ\beta\circ\alpha)^*)_{\bar\eta} : (R^1p_{1*}(\mathbb{Z}/l^n)_{\mathscr{A}})_{\bar\eta} \to (R^1p_{4*}(\mathbb{Z}/l^n)_{\mathscr{C}\times\mathscr{Y}})_{\bar\eta}\ .$$

Since we have an equivalence of categories between locally constant sheaves on U' and finite continuous $\pi_1(U'\bar\eta)$ modules we get that the morphism $R^1((\gamma\circ\beta\circ\alpha)^*)$ of sheaves induces the morphism $R^1((\gamma\circ\beta\circ\alpha)^*)_{\bar\eta}$ of $\pi_1(U',\bar\eta)$ modules. Now let us fix a point p_0 on Γ and consider the embedding

$$i : Y \to \Gamma\times Y$$

given by

$$y \mapsto (p_0, y)$$

this can be spread into a morphism from $\mathscr{Y}$ to $\mathscr{C}\times\mathscr{Y}$, then we have the morphism

$$R^1i^*_{\bar\eta} : H^1_{ét}(\Gamma\times Y,\mathbb{Q}_l) \to H^1_{ét}(Y,\mathbb{Q}_l)$$

of $\pi_1(U',\bar\eta)$ modules. Now by Poincare duality we get that

$$H^1_{ét}(Y,\mathbb{Q}_l) \cong \mathrm{Hom}_{\mathbb{Q}_l}(H^{2p-1}(Y,\mathbb{Q}_l(p-1)),\mathbb{Q}_l)$$

that is isomorphic to

$$H^{2p-1}_{ét}(Y,\mathbb{Q}_l(p-1))$$

and we get that the morphism

$$R^1i^*_{\bar\eta} : H^1_{ét}(\Gamma\times Y,\mathbb{Q}_l) \to H^{2p-1}_{ét}(Y,\mathbb{Q}_l(p-1))$$

is a morphism of $\pi_1(U',\bar\eta)$ modules and composing $R^1i^*_{\bar\eta}$ with $R^1((\gamma\circ\beta\circ\alpha)^*)_{\bar\eta}$ we get a morphism of $\pi_1(U',\bar\eta)$ modules from

$$H^1_{ét}(A,\mathbb{Q}_l) \to H^{2p-1}_{ét}(Y,\mathbb{Q}_l(1-p))$$

and tensoring both sides in the above we get that

$$H^1_{\acute{e}t}(A, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\acute{e}t}(Y, \mathbb{Q}_l)$$

which is a morphism of $\pi_1(U', \bar{\eta})$ modules.

Since the homomorphism w_* is compatible with the action of the fundamental group $\pi_1(U', \bar{\eta})$, this gives that the composition $w_* \circ \zeta_{\mathbb{Q}_l}$ is an injection of the $\pi_1(U', \bar{\eta})$-module $H^1_{\acute{e}t}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p))$ into the $\pi_1(U', \bar{\eta})$-module of vanishing cycles E. Let E_0 be the image of this injection.

Since U' is finite of degree n over U, the group $\pi_1(U', \bar{\eta})$ is a subgroup of finite index n in the étale fundamental group $\pi_1(U, \bar{\eta})$.

This is because of the following. Shrinking U we can assume that the morphism $U' \to U$ is smooth and also it is finite hence quasi-finite, therefore it is étale. Now take a Galois cover $Y \to U'$ then composing it with $U' \to U$ we get a finite étale, Galois cover $Y \to U$. Now the group $\mathrm{Aut}(Y|U')$ is a subgroup of $\mathrm{Aut}(Y|U)$ and since $Y \to U$ and $Y \to U'$ are Galois covers we get that

$$\mathrm{Aut}(Y|U') = m \ , \qquad \mathrm{Aut}(Y|U) = n \ .$$

And we have that $\mathrm{Aut}(Y|U')$ is a finite index subgroup of $\mathrm{Aut}(Y|U)$. Now we have to prove that $\pi_1(U', \bar{\eta})$ is a finite index subgroup of $\pi_1(U, \bar{\eta})$, to do that it is enough to prove that $\prod_Y \mathrm{Aut}(Y|U')$ is a finite index subgroup of $\prod_Y \mathrm{Aut}(Y|U)$. But this follows from the fact that $\mathrm{Aut}(Y|U')$ is a finite index subgroup of $\mathrm{Aut}(Y|U)$.

The group $\pi_1(U, \bar{\eta})$ acts continuously on E by the standard étale monodromy theory. Let us use the Picard-Lefschetz formula in order to show that E_0 is a $\pi_1^{\mathrm{tame}}(U, \bar{\eta})$-equivariant subspace in E. Obviously, it is enough to show that for each element $\gamma_s(u)$ in $\pi_1^{\mathrm{tame}}(U, \bar{\eta})$ and any element x in E_0 the element $\gamma_s(u)x$ is again in the space E_0.

Indeed, since $\langle \delta_s, \delta_s \rangle = 0$, the Picard-Lefschetz formula (4.7) and easy induction give that $(\gamma_s(u))^m x = x \pm m\bar{u}\langle x, \delta_s \rangle \delta_s$ for a natural number m, whence

$$\bar{u}\langle x, \delta_s \rangle \delta_s = \frac{1}{m}\left((\gamma_s(u))^m x \pm x\right) \ .$$

When m is the index of $\pi_1(U', \bar{\eta})$ in $\pi_1(U, \bar{\eta})$, then $(\gamma_s(u))^m$ sits in the subgroup $\pi_1(U', \bar{\eta})$, so that the right hand side of the latter formula is an element of E_0. Applying the Picard-Lefschetz formula again, we see that $\gamma_s(u)x$ is in E_0.

Let us give more detail on it. The group $\pi_1(U, \bar{\eta})$ acts tamely on $H^{2p-1}_{\acute{e}t}(\mathscr{Y}_{\bar{\eta}}, \mathbb{Q}_l)$, that is the action of the étale fundamental group factors through the action of the tame fundamental group $\pi_1^{\mathrm{tame}}(U, \bar{\eta})$. Now we have to show that E_0 is $\pi^{\mathrm{tame}}(U, \bar{\eta})$ equivariant. That is it is enough to show that for each element $\gamma_s(u)$ in $\pi_1^{\mathrm{tame}}(U, \bar{\eta})$ and any element x in E_0 the element $\gamma_s(u)(x)$ is again in E_0. By the Picard Lefschtez formulae we have that

$$(\gamma_s(u))^m x = x \pm m\bar{u}\langle x, \delta_s \rangle \delta_s \ .$$

Now since the fundamental group $\pi_1(U, \bar{\eta})$ acts tamely, we have for an element a in $\pi_1(U, \bar{\eta})$ such that its image under the natural homomorphism from $\pi_1(U, \bar{\eta})$ to $\pi_1^{\rm tame}(U, \bar{\eta})$ is $\gamma_s(u)$. Also we have that

$$a^m \cdot x = (\gamma_s(u))^m \cdot x$$

taking m to be index of $\pi_1(U', \bar{\eta})$ we get that a^m is in $\pi_1(U', \bar{\eta})$ and also we have that E_0 is $\pi_1(U', \bar{\eta})$ invariant. Therefore we get that

$$(\gamma_s(u))^m \cdot x - x$$

is in E_0, whence we get that E_0 is $\pi_1^{\rm tame}(U, \bar{\eta})$ invariant.

Thus, E_0 is a submodule in the $\pi_1^{\rm tame}(U, \bar{\eta})$-module E. Since E is known to be an absolutely irreducible (see, for example, Corollary 7.4 on page 249 in [12]), we see that either $E_0 = 0$ or $E_0 = E$. In the first case

$$H^1_{\acute{e}t}(A_{\bar{\eta},0}, \mathbb{Q}_l) = 0 \ ,$$

whence $A_{\bar{\eta},0} = 0$. In the second case

$$\zeta'_{\mathbb{Q}_l} : H^1_{\acute{e}t}(A_{\bar{\eta},0}, \mathbb{Q}_l(1-p)) \to H^1_{\acute{e}t}(A_{\bar{\eta},1}, \mathbb{Q}_l(1-p))$$

is an isomorphism, whence $A_{\bar{\eta},0} = A_{\bar{\eta},1}$. □

Let now

$$\mathscr{D}$$

be the discriminant variety of $\mathscr{X}$ in $\mathbb{P}^{m\vee}$, and let

$$T = \mathbb{P}^{m\vee} \smallsetminus \mathscr{D}$$

be the complement to $\mathscr{D}$ in $\mathbb{P}^{m\vee}$. Let

$$\mathscr{H}_T \to T$$

be the pull-back of the projection

$$p_2 : \mathscr{H} \to \mathbb{P}^{m\vee}$$

with respect to the embedding of T into $\mathbb{P}^{m\vee}$.

Recall that $\mathscr{Y} \to \mathscr{X}$ is the pull-back of the projection $p_1 : \mathscr{H} \to \mathbb{P}^m$ with respect to the embedding of $\mathscr{X}$ into $\mathbb{P}^m$ and $f : \mathscr{Y} \to \mathbb{P}^{m\vee}$ is the composition of the closed embedding of $\mathscr{Y}$ into $\mathscr{H}$ with the projection p_2.

For any closed point

$$t \in \mathbb{P}^{m\vee}$$

let

$$\mathscr{Y}_t$$

be the fibre of the morphism f, i.e. the intersection of the corresponding hyperplane H_t with $\mathscr{X}$. Let also

$$\mathscr{Y}_T$$

be the fibred product of $\mathscr{Y}$ and $\mathscr{H}_T$ over the universal hyperplane $\mathscr{H}$, and let $f_T : \mathscr{Y}_T \to T$ be the induced projection.

Let

$$S \subset T$$

be a Zariski open affine subset in T and let

$$U \subset U$$

be the c-open subset in S constructed exactly as in Section 4.4.

In other words, we define U by removing the images of the pull-backs of all closed embeddings into the model S_0 of S defined over the minimal field of definition of S.

Then U is a c-open subset in T, and in the dual projective space $(\mathbb{P}^m)^\vee$, such that, if

$$\xi$$

is the generic point of the projective space $\mathbb{P}^{m\vee}$ and $\bar{\xi}$ the corresponding geometric generic point, for any closed point $P \in U$ one has the isomorphism $\varkappa_P$ between $\mathscr{Y}_P$ and $\mathscr{Y}_{\bar{\xi}}$, and for any two closed points P and P' in U one has the scheme-theoretic isomorphism $\varkappa_{PP'}$ between $\mathscr{Y}_P$ and $\mathscr{Y}_{P'}$, constructed in Section 4.4.

In what follows we will be also assuming that Assumptions (A) and (B) are satisfied for the fibres $\mathscr{Y}_{\bar{\eta}}$ and $\mathscr{Y}_{\bar{\xi}}$. In particular, this guarantees that Assumptions (A) is also satisfied in a family, in the sense of the spreads of $A_{\bar{\eta}}$ and $A_{\bar{\xi}}$ over Zariski open subsets in some finite extensions of D or T. Proposition 4.4.5 would then guarantee that

$$\kappa_t(A_{t,0}) = A_{\bar{\xi},0}$$

for each close point t in U, and the same for Lefschetz pencils.

Theorem 4.5.2. *In the above terms and under the above assumptions, either* $A_{\bar{\xi},0} = 0$, *in which case* $A_{P,0} = 0$ *for each closed point* P *in* U, *or* $A_{\bar{\xi},0} = A_{\bar{\xi},1}$, *so that* $A_{P,0} = A_{P,1}$ *for any close point* P *in* U.

Proof. Indeed, let A be a Zariski closed subset in $\mathbb{P}^{m\vee}$, such that for each point t in the complement to A in $\mathbb{P}^{m\vee}$ the corresponding hyperplane H_t does not contain $\mathscr{X}$ and the scheme-theoretic intersection of $\mathscr{X}$ and H_t is either smooth or contains at most one singular point, which is double point. Let G be the Grassmannian of lines in $\mathbb{P}^{m\vee}$.

Let us prove that there is a Zariski open subset W in the Grassmannian G, such that for each line D in W the line D does not intersect A and the corresponding codimension 2 linear subspace in $\mathbb{P}^m$ intersects $\mathscr{X}$ transversally.

Indeed, since $\mathscr{X}$ inside $\mathbb{P}^m$ is a Lefschetz embedding, we have that codimension of A greater or equal than 2 in $\mathbb{P}^{m\vee}$, so the dimension of A is less than or equal to $m-2$. But suppose that every line intersects A, then the dimension theorem on the intersection of subvarieties of a projective space says that

$$\dim(A) + 1 - m \geq 0$$

that implies

$$\dim(A) \geq m - 1$$

which is a contradiction. So we have that there exists a line D in $\mathbb{P}^{m\vee}$ such that the intersection of D with A is empty.

On the other hand there exists a line D in $\mathbb{P}^{m\vee}$ such that the intersection of the corresponding codimension 2 subspace with $\mathscr{X}$ is transversal. This is because of the fact that, a general hyperplane section intersects $\mathscr{X}$ transversally. We need the following two lemmas.

Lemma 4.5.3. *Let $\mathscr{V}$ be a family of projective varieties parametrized by a projective algebraic variety B. Then the following set*

$$\{b \in B | V_b \cap X \neq \emptyset\}$$

is closed.

Proof. To prove this we consider the following diagrams.

$$\begin{array}{ccc} \mathbb{P}^n \times B & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{P}^n & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

We have the family $\mathscr{V}$ embedded inside $\mathbb{P}^n \times B$ and it is nothing but

$$V = \{(x, b) \in \mathbb{P}^n \times B | x \in V_b\} \ .$$

Now consider a Zariski closed subset X of $\mathbb{P}^n$ and consider the Cartesian squares as follows

$$\begin{array}{ccc} X \times B & \longrightarrow & \mathbb{P}^n \times B \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}^n \end{array}$$

$$\begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & \mathbb{P}^n \times B \end{array}$$

It means that Y is the fiber product of $X \times B$ and V over $\mathbb{P}^n \times B$, so the underlying topological space of Y is

$$\{(x, b) \in X \times B | x \in V_b\} = V \cap (X \times B)$$

Since $X \times B$ and V are closed inside $\mathbb{P}^n \times B$, we get that Y is closed. Now consider the projection

$$\pi : Y \to B$$

since it is the composition of

$$Y \to V, \quad V \to \mathbb{P}^n \times B \to B$$

and each of the above morphisms are proper we get that

$$\pi : Y \to B$$

is proper. Therefore we get that$\pi(Y)$ is closed in B. Now

$$\pi(Y) = \{b \in B | X \cap V_b \neq \emptyset\}$$

and that is closed. □

We apply the above lemma when B is G and X is A to get that all lines in G intersecting the closed set A forms a Zariski closed subset in G.

Lemma 4.5.4. *Let G be the Grassmanian of lines in $\mathbb{P}^{m\vee}$. Consider an ambient projective variety $\mathscr{Y}$, inside which the codimenison 2 subvarieties corresponding to the lines in G and $\mathscr{X}$ intersect. Then the set of all lines which intersect $\mathscr{X}$ non-transversally in $\mathscr{Y}$ is a closed set in G.*

Proof. Consider the following set

$$C = \{([a_0 : \cdots : a_m], ([b_0 : \cdots : b_m], [c_0 : \cdots : c_m])) \in \mathscr{X} \times G$$

$$| T_P(L_1 \cap L_2) + T_P \mathscr{X} \subset T_P \mathscr{Y}\}$$

here P is the point $[a_0 : \cdots : a_m]$ and L_1, L_2 denote the hyperplanes corresponding to the points $[b_0 : \cdots : b_m], [c_0 : \cdots : c_m]$ respectively.

We claim that this set is Zariski closed. For that we write down the equations of the codimension 2 subspace $L_1 \cap L_2$ corresponding to the line in $\mathbb{P}^{m\vee}$. A codimension 2 linear subspace in $\mathscr{Y}$ is defined by $\sum_i b_i x_i = 0$ and $\sum_j c_j x_j = 0$. Let the equation of $\mathscr{X}$ and $\mathscr{Y}$ be

$$f_1 = \cdots = f_l = 0$$

and

$$g_1 = \cdots = g_k = 0$$

respectively. Then the tangent space at $[a_0 : \cdots : a_m]$ is given by

$$\sum_i b_i x_i = 0, \quad \sum_j c_j x_j = 0 \ ,$$

$$\sum_i \frac{\partial f_r}{\partial x_i}(a_i)(x_i - a_i) = 0$$

for $r = 1, \cdots, l$ and

$$\sum_i \frac{\partial g_s}{\partial x_i}(a_i)(x_i - a_i) = 0$$

for $s = 1, \cdots, k$ to $L_1 \cap L_2, \mathscr{X}, \mathscr{Y}$ at $[a_0 : \cdots : a_m]$ respectively. Now consider the condition, that is

$$T_P(L_1 \cap L_2) + T_P\mathscr{X} \subset T_P\mathscr{Y}$$

this is given by the following equations. First write

$$\sum_i (b_i - c_i)x_i = 0$$

assume that $b_0 - c_0 \neq 0$, that gives us

$$x_0 = -\frac{\sum_{i \neq 0}(b_i - c_i)x_i}{b_0 - c_0}$$

and from the equation of the tangent space of $\mathscr{X}$ we get $(y_0, \cdots, y_m)$ such that $(y_0, \cdots, y_m)$ satisfies the equations of the tangent space to $\mathscr{X}$ at $[a_0 : \cdots : a_m]$. Then by the condition we have that

$$(x_0 + y_0, \cdots, x_m + y_m) \in T_P\mathscr{Y} \ .$$

Now put the value of x_0 that is given above in the equation of $T_P\mathscr{Y}$. Then we have the equation defining a Zariski closed subspace in some projective space. So the set C defined above is a Zariski closed set in some projective space. Consider the projection π from C to G, this is a closed set, since the projection π is proper. So $\pi(C)$ is a closed set that parametrizes all lines such that the corresponding codimension two subspace intersects $\mathscr{X}$ non-transversally in $\mathscr{Y}$. Therefore the complement to $\pi(C)$ gives us all lines in G parametrizing the lines such that the corresponding codimension 2 linear subspace intersects $\mathscr{X}$ transversally. □

Applying the above two lemmas (Lemma 4.5.4 being applied to the case when $\mathscr{Y} = \mathbb{P}^m$) we get that the set of lines W in G, such that the lines does not intersect A and the corresponding codimension 2 subspace intersect $\mathscr{X}$ transversally in $\mathbb{P}^m$ is a non-empty Zariski open subset of G. In other words, any line D from W gives rise to a Lefschetz pencil on $\mathscr{X}$.

Let Z be the complement to the above c-open subset U in $\mathbb{P}^{m\vee}$. Then Z is the union of a countable collection of Zariski closed irreducible subsets in $\mathbb{P}^{m\vee}$, each of which is irreducible. In particular, Z is c-closed. Let us show that it implies that the condition for a line $D \in G$ to be not a subset in Z is c-open.

For that we write Z as $\cup Z_i$. Now since a line D is irreducible, $D \subset Z$ is equivalent to the fact that D is a subset of Z_i for some i. Now we want to prove that the set of lines D in G, such that D is a subset of Z_i is closed. For that we consider the following subset C of

$$\mathbb{P}^{m\vee} \times G$$

consisting of pairs (t, D) such that $t \in D$ implies that $t \in Z_i$. Now we will prove that this set is Zariski closed. Considering the projection from C to the second factor we get the set of lines D such that D is a subset of Z_i, since the projection map is closed, this set will be closed. Now since considering a line D in $\mathbb{P}^m$, it is given by $m-1$ linear forms

$$L_0 = L_2 = \cdots = L_{m-2} = 0$$

where L_i is given by the equation

$$\sum_j a_{ij}x_j = 0$$

and the linear forms are linearly independent. where $[x_0 : \cdots : x_m]$ is a co-ordinate system in $\mathbb{P}^m$. Now a point $t = [t_0 : \cdots : t_m]$ belongs to D means that it satisfies the equations of D, that is

$$\sum_j a_{ij}t_j = 0$$

this can be written as a matrix equation. Since $L_1, \cdots, L_{m-2}$ are linearly independent we can write the above equations as

$$\sum_{j=0}^{m-2} a_{ij}t_j = a_{im-1}t_{m-1} + a_{im}t_m$$

and consider the matrix $A = (a_{ij})$ where $i, j = 0, \cdots, m-2$, A is invertible. Therefore we can write

$$t_j = A^{-1}(a_{im-1}t_{m-1} + a_{im}t_m)$$

that is

$$t_j = b_jt_{m-1} + b_{j+1}t_m$$

for each $j = 0, \cdots, m-2$. Now substituting t_j as above in the equation of Z_i we get the equations of C, hence C is Zariski closed. Therefore considering the second co-ordinate projection we get that the set of lines D such that $D \subset Z_i$ is Zariski closed. Therefore the set of lines D in G such that D is not a subset of Z is c-open.

The following lemma is simple but useful.

Lemma 4.5.5. *Let U_1, U_2 be two non-empty c-open subsets in an irreducible quasi-projective variety X. Then $U_1 \cap U_2$ is non-empty.*

Proof. Suppose if possible $U_1 \cap U_2 = \emptyset$ for two c-open sets U_1, U_2. We write X as $X \smallsetminus \emptyset$, that can be written as $X \smallsetminus (U_1 \cap U_2)$ that is same as $(X \smallsetminus U_1) \cup (X \smallsetminus U_2)$. Now $X \smallsetminus U_i$ is a countable union of Zariski closed subsets in X, each of which is not the whole of X. Then the above union $(X \smallsetminus U_1) \cup (X \smallsetminus U_2)$ is a countable union of Zariski closed subsets in X each of which is not the whole of X. Since the ground field k is uncountable, this contradicts Lemma 4.2.1. $\square$

By Lemma 4.5.5, the intersection of the corresponding c-open subset in G with W is non-empty, so that we can choose a line D, such that D gives a Lefschetz pencil

$$f_D : \mathscr{Y}_D \to D$$

and

$$D \cap U \neq \emptyset \ .$$

By the same lemma, the intersection of the two c-open subsets $D \cap U$ and U_D in the line D is nonempty,

$$D \cap U \cap U_D \neq \emptyset \ .$$

Let now P_0 be a point in $D \cap U$ and let $\bar{\eta}$ be the geometric generic point of D. By Proposition 4.5.1, either $A_{\bar{\eta},0} = 0$ or $A_{\bar{\eta},0} = A_{\bar{\eta},1}$.

Suppose $A_{\bar{\eta},0} = 0$. Proposition 4.4.5, being applied to the pencil f_D, gives that $A_{P_0,0} = 0$. Applying the same proposition to the family $f : \mathscr{Y} \to \mathbb{P}^{m\vee}$ we obtain that $A_{\bar{\xi},0} = 0$ and so for each closed point P in U the abelian variety $A_{P,0}$ is zero. Similarly, if $A_{\bar{\eta},0} = A_{\bar{\eta},1}$ then, by Proposition 4.4.5 and Remark 4.4.6 applied to f_D we obtain that $A_{P_0,0} = A_{P_0,1}$. Applying Proposition 4.4.5 and Remark 4.4.6 to the family f we see that

$$A_{\bar{\xi},0} = A_{\bar{\xi},1}$$

and

$$A_{P,0} = A_{P,1}$$

for each closed point P in U. $\square$

4.6 The constructibility result for cycles on hyperplane section

We keep the notation and assumptions of the previous section. Also we shall keep assuming that p is either 1 or 2, i.e. $\mathscr{X}$ is either a surface or a fourfold in $\mathbb{P}^m$, so that Proposition 4.5.1 works. Recall that T is a Zariski open subset in $\mathbb{P}^{m\vee}$, such that $\mathscr{Y}_t$ is smooth for each closed point $t \in T$. Suppose that Assumptions (A) and (B) hold for the geometric generic fibre $\mathscr{Y}_{\bar{\xi}}$. Removing a finite number of

Zariski closed subsets from T if necessary, we may assume that the assumptions hold for each closed point t in T.

Let

$$T^{\natural}$$

be the set of closed points in T such that

$$t \in T^{\natural} \Leftrightarrow A_{t,0} = A_{t,1} \ .$$

For simplicity, in this section we assume that

$$H^{2p+1}_{ét}(\mathscr{X}, \mathbb{Q}_l) = 0 \ .$$

This gives that $A_{t,1} = A_t$ for each closed point t in T, so that

$$t \in T^{\natural} \Leftrightarrow r_{t*} = 0 \ .$$

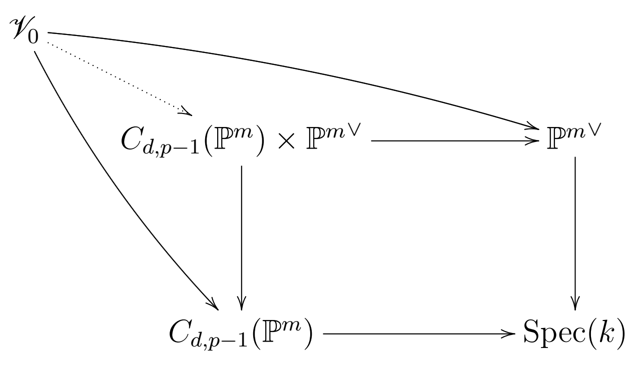
where r_{t*} is, as usual, the proper push-forward from $A^p(\mathscr{Y}_t)$ to $A^{p+1}(\mathscr{X})$. Since the group $A^p(\mathscr{Y}_{\bar{\xi}})$ is weakly representable, we can choose a smooth projective curve C over $\bar{\xi}$ and an appropriate algebraic cycle Z on $C \times \mathscr{Y}_{\bar{\xi}}$, such that the induced homomorphism Z_* from $A^1(C)$ to $A^p(\mathscr{Y}_{\bar{\xi}})$ is surjective. Then the homomorphism θ^p_d from $C^p_{d,d}(\mathscr{Y}_{\bar{\xi}})$ to $A^p(\mathscr{Y}_{\bar{\xi}})$ is surjective, where d is the genus of the curve C (see the proof of Proposition 4.2.4). Shrinking T further we may assume that the homomorphism θ^p_d from $C^p_{d,d}(\mathscr{Y}_t)$ to $A^p(\mathscr{Y}_t)$ is surjective for each close point t in the Zariski open subset T of the dual space $\mathbb{P}^{m\vee}$.

Proposition 4.6.1. *Under the above assumptions, the set $T^{\natural}$ is constructible.*

Proof. Consider the set

$$\mathscr{V} = \{(Z,t) \in \mathscr{C}^{p+1}_d(\mathscr{X}) \times \mathbb{P}^{m\vee} \mid Z \subset H_t\} \ ,$$

where $Z \subset H_t$ means that the codimension $p+1$ algebraic cycle Z of degree d on $\mathscr{X}$ is supported on the hyperplane section $\mathscr{Y}_t = \mathscr{X} \cap H_t$ for t in $\mathbb{P}^{m\vee}$. Let us prove that $\mathscr{V}$ is Zariski closed. For that we use the Cartesian squares



and

$$\begin{array}{ccc} \mathscr{V} & \longrightarrow & \mathscr{V}_0 \\ \downarrow & & \downarrow \\ C_{d,p-1}(\mathscr{X}) & \longrightarrow & C_{d,p-1}(\mathbb{P}^m) \end{array}$$

Here $C_{d,p-1}(\mathscr{X})$ stands for the Chow variety parametrizing the dimension $p-1$, degree d algebraic cycles on $\mathscr{X}$, and $\mathscr{V}_0$ is defined as

$$\{(Z,t) \in C_{d,p-1}(\mathbb{P}^m) \times \mathbb{P}^{m\vee} | Z \subset H_t\}$$

where $Z \subset H_t$ means that the support of Z is contained in H_t.

So from this two diagrams it is clear that $\mathscr{V}$ is the Cartesian square of the incidence subvariety $\mathscr{V}_0$ and $C_{d,p-1}(\mathscr{X})$ over $C_{d,p-1}(\mathbb{P}^m)$. Since we know that the Cartesian product of two closed subschemes is closed, so $\mathscr{V}$ will be closed once we prove that $\mathscr{V}_0$ is closed.

Therefore our aim is to prove that $\mathscr{V}_0$ is closed. Let $[x_0 : \cdots : x_m]$ be a point in $\cup_i V_i$, since $\cup_i V_i$ is a subset of H_t we have that $[x_0 : \cdots : x_m]$ satisfies the equation of H_t. So we write out the equation of the hyperplane H_t corresponding to t, that is given by the co-ordinates $[t_0 : \cdots : t_m]$, and the equation of the hyperplane is

$$\sum_i t_0 x_0 = 0 \,.$$

From this we can write

$$x_0 = \frac{-\sum_{i\neq 0} t_i x_i}{t_0} \,.$$

Let $f_1^Z = \cdots = f_k^Z = 0$ be the equations of the support of an algebraic cycle Z of dimension $p-1$ and of degree d. Then we have

$$f_1^Z([x_0 : \cdots : x_m]) = \cdots = f_k^Z[x_0 : \cdots : x_m] = 0 \,.$$

Substituting the value of x_0 in the above equations of $\cup_j V_j$, we get that

$$f_1^Z\left(\left[\frac{-\sum_{i\neq 0} t_i x_i}{t_0} : \cdots : x_m\right]\right) = \cdots = f_k^Z\left(\left[\frac{-\sum_{i\neq 0} t_i x_i}{t_0} : \cdots : x_m\right]\right) = 0$$

which is same as saying that

$$f_1^Z\left(\left[-\sum_{i\neq 0} t_i x_i : t_0 x_1 : \cdots : t_0 x_m\right]\right) = \cdots = f_k^Z\left(\left[-\sum_{i\neq 0} t_i x_i : t_0 x_1 : \cdots : t_0 x_m\right]\right) = 0 \,.$$

So we get a system of equations defining $\mathscr{V}_0$, this proves that it is a Zariski closed subset in $C_{d,p-1}(\mathbb{P}^m) \times \mathbb{P}^{m\vee}$. This proves that $\mathscr{V}$ is a Zariski closed susbet in $C_{d,p-1}(\mathscr{X}) \times \mathbb{P}^{m\vee}$.

Let $v_T : \mathscr{V}_T \to T$ be the corresponding pull-back of the projection to $\mathbb{P}^{m\vee}$ with respect to the inclusion of T into $\mathbb{P}^{m\vee}$.

We have a natural morphism from $\mathscr{V}_0$ to $\mathbb{P}^{m\vee}$ and also we have the embedding of T in $\mathbb{P}^{m\vee}$. So we can form the fibred product

$$\begin{array}{ccc} \mathscr{V}_{0T} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathscr{V}_0 & \longrightarrow & \mathbb{P}^{m\vee} \end{array}$$

and the fibred product

$$\begin{array}{ccc} \mathscr{V}_{T} & \longrightarrow & \mathscr{V}_{0T} \\ \downarrow & & \downarrow \\ \mathscr{V} & \longrightarrow & \mathscr{V}_0 \end{array}$$

Composing the above two diagrams we get the fiber square

$$\begin{array}{ccc} \mathscr{V}_{T} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathscr{V} & \longrightarrow & \mathbb{P}^{m\vee} \end{array}$$

The morphism $\mathscr{V}_T \to T$ is denoted as v_T and the composition

$$\mathscr{V}_T \to V \to C_{d,p-1}(\mathscr{X})$$

is denoted as s_T. In terms of codimension, s_T is the morphism from $\mathscr{V}_T$ to $C^{p+1}_{d,d}(\mathscr{X})$.

Now we have the following Cartesian square

$$\begin{array}{ccc} \mathscr{V}_{T} & \longrightarrow & C_{d,p-1}(\mathscr{X}) \\ \downarrow & & \downarrow \\ C_{d,p-1}(\mathscr{X}) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

that gives us a unique morphism from $\mathscr{V}_T$ to $C_{d,p-1}(\mathscr{X}) \times C_{d,p-1}(\mathscr{X})$, therefore composing this morphism with $\mathscr{V}_T \times \mathscr{V}_T$ to $\mathscr{V}_T$ we get the morphism s_T^2. If $\mathscr{V}_T^2$ is the 2-fold fibred product of $\mathscr{V}_T$ over T, and we then consider the corresponding morphisms v_T^2 from $\mathscr{V}_T^2$ to T and, as above, s_T^2 from $\mathscr{V}_T^2$ to $C_{d,d}^{p+1}(\mathscr{X})$, we obtain the diagram

$$\begin{array}{ccccc} \mathscr{V}_T^2 & \xrightarrow{s_T^2} & C_{d,d}^{p+1}(\mathscr{X}) & \xrightarrow{\theta_d^{p+1}} & A^{p+1}(\mathscr{X}) \\ \big\downarrow{\scriptstyle v_T^2} & & & & \\ T & & & & \end{array} \tag{4.8}$$

By Corollary 4.1.2 we have that $(\theta_d^{p+1})^{-1}(0)$ is the union of a countable collection of irreducible Zariski closed subsets in $C_{d,d}^{p+1}(\mathscr{X})$, say $(\theta_d^{p+1})^{-1}(0) = \cup_{i\in I} Z_i$. Let $W_i = (s_T^2)^{-1}(Z_i)$ for each $i \in I$. For any closed point t in T the pre-image $(v_T^2)^{-1}(t)$ is the 2-fold product $\mathscr{V}_t^2$ of the fibre $\mathscr{V}_t$ of the morphism v_T at t over $\mathrm{Spec}(k)$.

Since the homomorphism θ_d^p from $C_{d,d}^p(\mathscr{Y}_t)$ to $A^p(\mathscr{Y}_t)$ is surjective, we obtain that the condition $r_{t*} = 0$ is equivalent to the condition that the fibre $\mathscr{V}_t^2$ of the morphism v_T^2 at t is a subset of the pre-image $\cup_{i\in I} W_i$ of 0 under the composition $\theta_d^{p+1} \circ s_T^2$. Let us explain this point in more details. By definition the set underlying the fiber product $\mathscr{V}_t^2$ is

$$\{(Z_1, Z_2) | Z_1, Z_2 \subset H_t\}$$

the morphism s_T^2 is nothing but

$$(Z_1, Z_2, t) \mapsto (Z_1, Z_2) \ .$$

Suppose that $r_{t*} = 0$, and take (Z_1, Z_2) in $\mathscr{V}_t^2$. Then Z_1, Z_2 are supported on H_t. Now consider the cycle class $[Z_1 - Z_2]$, then

$$r_{t*}[Z_1 - Z_2] = 0 = \theta_d^{p+1} \circ r_{t*}(Z_1, Z_2) = 0$$

by the commutativity of the following diagram.

$$\begin{array}{ccc} C_{d,d}^p(\mathscr{X} \cap H_t) & \xrightarrow{r_{t*}} & C_{d,d}^{p+1}(\mathscr{X}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \theta_d^{p+1}} \\ A^p(\mathscr{X} \cap H_t) & \xrightarrow{r_{t*}} & A^{p+1}(\mathscr{X}) \end{array}$$

Therefore it follows that (Z_1, Z_2) is an element of $\cup_i W_i$, so

$$\mathscr{V}_t^2 \subset \cup_i W_i \ .$$

Now suppose that $\mathscr{V}_t^2$ is a subset of $\cup_i W_i$, so take an element (Z_1, Z_2) inside $\mathscr{V}_t^2$, then it is in W_i for some i. That would imply that $(s_T(Z_1), s_T(Z_2))$ is in $(\theta_d^{p+1})^{-1}(0)$. So we get that

$$[s_T(Z_1) - s_T(Z_2)] = 0$$

that is same as saying, by the commutativity of the above diagram that

$$r_{t*}[Z_1 - Z_2] = 0$$

since θ_d^p is surjective from $C^p_{d,d}(\mathscr{X} \cap H_t)$ to $A^p(\mathscr{X} \cap H_t)$ we get that

$$r_{t*} = 0 \ .$$

Thus, the condition $r_{t*} = 0$ is equivalent to the condition that the fibre $\mathscr{V}_t^2$ is a subset in the pre-image $\cup_{i\in I} W_i$ of 0 under the composition $\theta_d^{p+1} \circ s_T^2$. By Lemma 4.2.1, this is equivalent to say that $\mathscr{V}_t^2$ is a subset in $W_{i_1} \cup \cdots \cup W_{i_n}$ for a finite collection of indices $i_1, \ldots, i_n$ in I.

Lemma 4.6.2. *Let $\mathscr{V}$ be a family of projective varieties parametrized by a projective algebraic variety B. Then the following set*

$$\{b \in B | V_b \subset X\}$$

is constructible.

Proof. To prove this we consider the following diagrams.

$$\begin{array}{ccc} \mathbb{P}^n \times B & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{P}^n & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

We have the family $\mathscr{V}$ embedded inside $\mathbb{P}^n \times B$ and it is nothing but

$$V = \{(x, b) \in \mathbb{P}^n \times B | x \in V_b\} \ .$$

Now consider a Zariski closed subset X of $\mathbb{P}^n$ and consider the Cartesian squares

$$\begin{array}{ccc} X \times B & \longrightarrow & \mathbb{P}^n \times B \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{P}^n \end{array}$$

and

$$\begin{array}{ccc} Y & \longrightarrow & V \\ \downarrow & & \downarrow \\ X \times B & \longrightarrow & \mathbb{P}^n \times B \end{array}$$

It means that Y is the fiber product of $X \times B$ and V over $\mathbb{P}^n \times B$, so the underlying topological space of Y is

$$\{(x,b) \in X \times B | x \in V_b\} = V \cap (X \times B)$$

Since $X \times B$ and V are closed inside $\mathbb{P}^n \times B$, we get that Y is closed. Now consider the projection

$$\pi : Y \to B$$

since it is the composition of

$$Y \to V \ , \quad V \to \mathbb{P}^n \times B \to B$$

and each of the above morphisms are proper we get that

$$\pi : Y \to B$$

is proper. Therefore we get that $\pi(Y)$ is closed in B.

Now

$$\pi(Y) = \{b \in B | X \cap V_b \neq \emptyset\}$$

and we can describe the set

$$\{b \in B | V_b \subset X\}$$

as

$$\pi(Y) \cap (B \smallsetminus \pi(Y'))$$

where

$$Y' = \{(x,b) \in Z \times B | x \in V_b\}$$

and Z is the complement of X in $\mathbb{P}^n$. So we have

$$\pi(Y') = \{b \in B | Z \cap V_b \neq \emptyset\} \ .$$

So arguing as before we prove that $\pi(Y')$ is closed. Now we prove that the set

$$\{b \in B | V_b \subset X\}$$

is indeed the set

$$\pi(Y) \cap (B \smallsetminus \pi(Y')) \ .$$

So let $b \in B$ be in the above intersection, then it means that

$$V_b \cap X \neq \emptyset, \quad V_b \cap Z = \emptyset$$

it means that the intersection of complement of X with V_b is empty so we have that

$$V_b \subset X \ .$$

Now if $V_b \subset X$ then we have $V_b \cap Z = \emptyset$. Therefore the set

$$\{b \in B \mid V_b \subset X\}$$

is constructible. $\square$

Now we consider the family $\mathscr{V}^2$ over the base $\mathbb{P}^{m\vee}$ and apply Lemma 4.6.2 with X to be equal to

$$W_{i_1} \cup \cdots \cup W_{i_n} \ .$$

Then we get

$$\{t \in \mathbb{P}^{m\vee} \mid \mathscr{V}_t^2 \subset X\}$$

is a constructible set. $\square$

We will also need the following lemma.

Lemma 4.6.3. *Let V be an irreducible quasi-projective variety over k, and let U be a nonempty c-open subset in V. Then the Zariski closure of U in V is V.*

Proof. Indeed, since U is c-open, there exists a countable union $Z = \cup_{i \in I} Z_i$ of Zariski closed irreducible subsets in V, such that $U = V \smallsetminus Z$. Then $\bar{U}$ is nothing but the complement to the interior $\mathrm{Int}(Z)$ of the set Z in V. Assume that that $\mathrm{Int}(Z)$ is nonempty. Then there exists a nonempty subset W in $\mathrm{Int}(Z)$, which is Zariski open in V. By Lemma 4.2.1, there exists an index $i_0 \in I$, such that W is contained in Z_{i_0}. This gives that $\mathrm{Int}(Z_{i_0})$ of the set Z_{i_0} is nonempty. This is not possible as Z_{i_0} is a closed proper subset in a Zariski topological space. $\square$

4.7 The Gysin hyperplane section map on Chow groups

Recall that we assume that Assumption (A) is satisfied in a family for the fibres of the morphism $f_T : \mathscr{Y}_T \to T$, in the sense of Section 4.4. The main result of the thesis is the following theorem, see [4].

Theorem 4.7.1. *Assume, in addition, that, if $\mathscr{Y}_t$ has one ordinary double point, the group $A^p(\tilde{\mathscr{Y}_t})$ is weakly representable, where $\tilde{\mathscr{Y}_t}$ is a resolution of $\mathscr{Y}_t$, the group $A^{p+1}(\mathscr{Y}_t)$ is weakly representable for each nonsingular $\mathscr{Y}_t$ and $A^{p+1}(\mathscr{X})$ is not rationally weakly representable. Then the kernel of the push-forward homomorphism from $A^p(\mathscr{Y}_t)$ to $A^{p+1}(\mathscr{X})$ is countable for each t of the c-open subset U in $\mathbb{P}^{m\vee}$.*

Proof. The proof of this theorem is somewhat long and occupies the rest of this section. As the proof is long we divide it into many steps.

By Theorem 4.5.2, we have that, either $A_{\bar{\xi},0} = 0$ or $A_{\bar{\xi},0} = A_{\bar{\xi}}$. Suppose the latest. By the same Theorem 4.5.2, $A_{t,0} = A_t$ for each closed point t in the c-open subset U in T. On the other hand, U is a subset in $T^\natural$, and the set $T^\natural$ is constructible by Proposition 4.6.1. With this assumption in hand let's first show that $A_{t,0} = A_t$ for all t in a Zariski open subset of T.

Step I :

Represent U as the complement to the countable union $\cup_{i \in I} D_i$ of irreducible Zariski closed subsets D_i in T. Also we write $T^\natural$ as the countable union $\cup_{j \in J} T_j^\natural$, where $T_j^\natural$ is Zariski open in an irreducible Zariski closed subset Z_j in T.

Let Z be the union $\cup_{j \in J} Z_j$ and let W be the complement to Z in T. Note that W is c-open in T and $W \cap U = \emptyset$. The intersection of W and U is the complement to the union of all D_i and Z_j, $i \in I$, $j \in J$, in T. As $U \neq \emptyset$, it follows that $D_i \neq T$ for each index i. Since $W \cap U = \emptyset$, by Lemma 4.2.1, there must exist an index $j_0 \in J$, such that

$$Z_{j_0} = T \ .$$

This gives us $A_{t,0} = A_t$, i.e. $r_{t*} = 0$, for each closed point t in the nonempty Zariski open subset $T_{j_0}^\natural$ in T.

By Lemma 4.6.3, the intersection of $T^\natural$ with U is nonempty. Let $f_D : \mathscr{Y}_D \to D$ be a Lefschetz pencil for $\mathscr{X}$, such that the set-theoretic intersection of the line $D = \mathbb{P}^1$ with the set $T^\natural \cap U$ is nonempty. By using the fact that, the group $A^{p+1}(\mathscr{Y}_t)$ is weakly representable for each $t \in T$ and D passes through U, it follows that the group $A^{p+1}(\mathscr{Y}_{\bar{\eta}})$ is weakly representable too. This we prove in detail. This consists of step II.

Step II :

Indeed, let D be a Lefschetz pencil such that $D \cap U$ is non-empty. Let U_D be the c-open subset of D consisting of closed points t of D, such that $\mathscr{Y}_t \cong \mathscr{Y}_{\bar{\eta}}$, where the isomorphism is over $\bar{\eta}$ and $\bar{\eta}$ is the geometric generic point of D. Since $D \cap U$ is c-open in D and U_D is c-open in D we have $D \cap U \cap U_D \neq \emptyset$. Therefore for a closed point t in the above intersection, we have

$$\mathscr{Y}_t \cong \mathscr{Y}_{\bar{\eta}} \ .$$

Note that for any closed point t in T, $A^{p+1}(\mathscr{Y}_t)$ is weakly representable. Now we have to prove that $A^{p+1}(\mathscr{Y}_{\bar{\eta}})$ is weakly representable. So we choose a point t in $D \cap U \cap U_D$, so that $\mathscr{Y}_t$ and $\mathscr{Y}_{\bar{\eta}}$ are isomorphic as schemes over $\bar{\eta}$. So we have the following fiber square.

$$\begin{array}{ccc} \mathscr{Y}_{\bar{\eta}} & \longrightarrow & \bar{\eta} \\ \downarrow & & \downarrow \\ \mathscr{Y}_t & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Since $A^{p+1}(\mathscr{Y}_t)$ is weakly representable, there exists a smooth projective curve Γ_t, and an algebraic cycle Z_t supported on $\Gamma_t \times \mathscr{Y}_t$ such that the homomorphism Z_{t*} from $A^1(\Gamma_t)$ to $A^{p+1}(\mathscr{Y}_t)$ is onto.

Since for each closed point t in the c-open set U we have the scheme-theoretical isomorphism between $\mathscr{Y}_t$ and $\mathscr{Y}_{\bar{\eta}}$, the group $A^{p+1}(\mathscr{Y}_{\bar{\eta}})$ is also weakly representable, and we can chose a nonsingular projective curve $\Gamma_{\bar{\eta}}$ and an algebraic cycle $\mathscr{Z}$ on $\Gamma_{\bar{\eta}} \times_{\bar{\eta}} \mathscr{Y}_{\bar{\eta}}$, such that Γ_t and Z_t will be specializations of $\Gamma_{\bar{\eta}}$ and $\mathscr{Z}_{\bar{\eta}}$ at t. Let t be in the c-open set U, then the following diagram will be commutative

$$\begin{array}{ccc} \Gamma_{\bar{\eta}} & \longrightarrow & \bar{\eta} \\ \downarrow & & \downarrow \\ \Gamma_t & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

and let the correspondence $Z_{\bar{\eta}}$ supported on $\Gamma_{\bar{\eta}} \times \mathscr{Y}_{\bar{\eta}}$ be defined to be the pullback of Z_t under the isomorphism

$$\varkappa : \Gamma_{\bar{\eta}} \times \mathscr{Y}_{\bar{\eta}} \to \Gamma_t \times \mathscr{Y}_t \ .$$

We denote the isomorphism from $\Gamma_{\bar{\eta}}$ to Γ_t by $\varkappa_1$ and the isomorphism from $\mathscr{Y}_{\bar{\eta}}$ to $\mathscr{Y}_t$ as $\varkappa_2$. Having this, we have to prove that the homomorphism $Z_{\bar{\eta}*}$ from $A^1(\Gamma_{\bar{\eta}})$ to $A^{p+1}(\mathscr{Y}_{\bar{\eta}})$ is onto. To do that, we prove the following diagram is commutative.

$$\begin{array}{ccc} A^1(\Gamma_t) & \xrightarrow{Z_{t*}} & A^{p+1}(\mathscr{Y}_t) \\ \downarrow{\scriptstyle \varkappa_1 *} & & \downarrow{\scriptstyle \varkappa_2^*} \\ A^1(\Gamma_{\bar{\eta}}) & \xrightarrow{Z_{\bar{\eta}*}} & A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \end{array}$$

That is

$$\varkappa_2^* \circ Z_{t*} = Z_{\bar{\eta}*} \circ \varkappa_1^* \ .$$

So we write out the formula for

$$Z_{\bar{\eta}*} \circ \varkappa_1^*(z) = (\mathrm{pr}_{\mathscr{Y}_{\bar{\eta}}*}(Z_{\bar{\eta}} \cdot \mathrm{pr}_{\Gamma_{\bar{\eta}}}^* \circ \varkappa_1^*(z)))$$

and consider the following commutative diagram.

$$\begin{array}{ccc} \Gamma_{\bar{\eta}} \times \mathscr{Y}_{\bar{\eta}} & \xrightarrow{\mathrm{pr}_{\Gamma_{\bar{\eta}}}} & \Gamma_{\bar{\eta}} \\ \downarrow{\scriptstyle \varkappa} & & \downarrow{\scriptstyle \varkappa_1} \\ \Gamma_t \times \mathscr{Y}_t & \xrightarrow{\mathrm{pr}_{\Gamma_t}} & \Gamma_t \end{array}$$

This gives

$$\mathrm{pr}_{\Gamma_{\bar\eta}}^* \circ \varkappa_1^* = \varkappa^* \circ \mathrm{pr}_{\Gamma_t}^* \ ,$$

and hence we can conclude from the above that

$$Z_{\bar\eta *} \circ \varkappa_1^*(z) = (\mathrm{pr}_{\mathscr{Y}_{\bar\eta} *}(Z_{\bar\eta} \cdot \varkappa^* \circ \mathrm{pr}_{\Gamma_t}^*(z)) \ .$$

This right hand side in the above expression is equal to

$$(\mathrm{pr}_{\mathscr{Y}_{\bar\eta} *}(\varkappa^*(Z_t) \cdot \varkappa^* \circ \mathrm{pr}_{\Gamma_t}^*(z)) \ .$$

Since $\varkappa^*$ is a group homomorphism we have that the above is equal to

$$\mathrm{pr}_{\mathscr{Y}_{\bar\eta} *} \circ \varkappa^*(Z_t \cdot \mathrm{pr}_{\Gamma_t}^*(z)) \ .$$

Let us consider the following fiber square

$$\begin{array}{ccc} \Gamma_{\bar\eta} \times \mathscr{Y}_{\bar\eta} & \xrightarrow{\mathrm{pr}_{\mathscr{Y}_{\bar\eta}}} & \mathscr{Y}_{\bar\eta} \\ \Big\downarrow{\scriptstyle \varkappa} & & \Big\downarrow{\scriptstyle \varkappa_2} \\ \Gamma_t \times \mathscr{Y}_t & \xrightarrow{\mathrm{pr}_{\mathscr{Y}_t}} & \mathscr{Y}_t \end{array}$$

where $\varkappa$ is flat and $\mathrm{pr}_{\mathscr{Y}_{\bar\eta}}$ is proper. By the well-known property of algebraic cycles (see Proposition 1.7 on page 18 in [13] or Section 2 above) we have that

$$\mathrm{pr}_{\mathscr{Y}_{\bar\eta} *} \circ \varkappa^* = \varkappa_2^* \circ \mathrm{pr}_{\mathscr{Y}_t *} \ .$$

Thus we have

$$Z_{\bar\eta *} \circ \varkappa_1^* = \varkappa_2^* \circ Z_{t*} \ ,$$

and therefore the diagram

$$\begin{array}{ccc} A^1(\Gamma_t) & \xrightarrow{Z_{t*}} & A^{p+1}(\mathscr{Y}_t) \\ \Big\downarrow{\scriptstyle \varkappa_{1*}} & & \Big\downarrow{\scriptstyle \varkappa_2^*} \\ A^1(\Gamma_{\bar\eta}) & \xrightarrow{Z_{\bar\eta *}} & A^{p+1}(\mathscr{Y}_{\bar\eta}) \end{array}$$

is commutative, and observing that the vertical homomorphisms are isomorphisms and the top horizontal morphism is onto, weget $Z_{\bar\eta *}$ is onto. So the group $A^{p+1}(\mathscr{Y}_{\bar\eta})$ is weakly representable.

The third step consists of spreading out $\Gamma_{\bar\eta}$ and $Z_{\bar\eta}$ and the effects of this spreading on the Chow groups.

Step III :

Let D' be a finite extension of the curve D, such that both $\Gamma_{\bar{\eta}}$ and $Z_{\bar{\eta}}$ are defined over the function field $k(D')$. Spreading out the curve $\Gamma_{\bar{\eta}}$ and the cycle $Z_{\bar{\eta}}$ we obtain a relative curve

$$\mathscr{G} \to V'$$

and a relative cycle

$$\mathscr{Z}$$

on $\mathscr{G} \times_{V'} \mathscr{Y}_{V'}$ over the preimage V' of a certain Zariski open subset V in D under the map $D' \to D$.

We also obtain a homomorphism

$$\mathscr{Z}_* : A^1(\mathscr{G}) \to A^{p+1}(\mathscr{Y}_{V'}) \ ,$$

such that, by definition,

$$\mathscr{Z}_*(a) = \mathrm{pr}_{\mathscr{Y}_{V'} *}(\mathscr{Z} \cdot \mathrm{pr}_{\mathscr{G}}^*(z)) \ .$$

To be more precise, to check whether the homomorphism $\mathscr{Z}$ is well defined we have to check that the morphism is proper or not. Since proper morphisms are stable under base change. It is enough to show that the morphism $\mathscr{G} \to V'$ is proper. Now consider the morphism $V' \to D'$, being an open immersion, it is separated.

Consider the composition

$$\mathscr{G} \to V' \to D' \ .$$

We want to prove that the morphism $\mathscr{G} \to D'$ is proper. Since $\mathscr{G}_{\bar{\eta}}$ is the smooth projective curve $\Gamma_{\bar{\eta}}$, we get that the morphism $\mathscr{G}$ to D' is locally projective, hence it is proper. On the other hand the open immersion $V' \to D'$ is separated, therefore we get that the morphism $\mathscr{G} \to V'$ is proper. So we can indeed define the homomorphism $\mathscr{Z}_*$.

Next, compactifying and resolving singularities, we obtain a surface $\mathscr{G}'$, a codimension 1 algebraic cycle $\mathscr{Z}'$ on the variety $\mathscr{G}' \times_{D'} \mathscr{Y}_{D'}$ and the homomorphism $\mathscr{Z}'_*$ from $A^1(\mathscr{G}')$ to $A^{p+1}(\mathscr{Y}_{D'})$. Notice that the following diagram is commutative:

$$\begin{array}{ccc} A^1(\mathscr{G}') & \xrightarrow{\mathscr{Z}'_*} & A^{p+1}(\mathscr{Y}_{D'}) \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle g^*} \\ A^1(\mathscr{G}'_{\eta'}) & \xrightarrow{\mathscr{Z}'_{\eta' *}} & A^{p+1}(\mathscr{Y}_{\eta'}) \end{array} \tag{4.9}$$

where $\mathscr{G}'_{\eta'}$ is $\Gamma_{\eta'}$. Indeed, take a in $A^1(\mathscr{G}')$. Then $g^* \mathscr{Z}'_*(a)$ is by definition equal to

$$g^*(\mathrm{pr}_{\mathscr{Y}_{D'} *}(\mathscr{Z}' \cdot \mathrm{pr}_{\mathscr{G}'}^*(a))) \ .$$

Consider the fiber square

$$\begin{array}{ccc}
\mathscr{Y}_{\eta'} \times \mathscr{G}'_{\eta'} & \xrightarrow{g\times f} & \mathscr{Y}_{D'} \times \mathscr{G}' \\
\Big\downarrow{\scriptstyle \mathrm{pr}_{\mathscr{Y}_{\eta'}}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\mathscr{Y}_{D'}}} \\
\mathscr{Y}_{\eta'} & \xrightarrow{g} & \mathscr{Y}_{D'}
\end{array}$$

The morphism $\mathrm{pr}_{\mathscr{Y}_{D'}}$ is proper and $g \times f$ is flat. Therefore by the formula present on page 9, we obtain

$$g^* \circ \mathrm{pr}_{\mathscr{Y}_{D'}*} = \mathrm{pr}_{\mathscr{Y}_{\eta'}*} \circ (g \times f)^* \ .$$

It follows from the above formula that

$$g^*(\mathrm{pr}_{\mathscr{Y}_{D'}*}(\mathscr{Z}' \cdot \mathrm{pr}^*_{\mathscr{G}'}(a)))$$

is equal to

$$\mathrm{pr}_{\mathscr{Y}_{\eta'}*} \circ (g \times f)^*(\mathscr{Z}' \cdot \mathrm{pr}^*_{\mathscr{G}'}(a)) \ .$$

Since $(g \times f)^*$ is a ring homomorphism, the above is same as

$$\mathrm{pr}_{\mathscr{Y}_{\eta'}*}((g \times f)^*(\mathscr{Z}') \cdot (g \times f)^* \mathrm{pr}^*_{\mathscr{G}'}(a)) \ .$$

Now on one hand we have

$$(g \times f)^*(\mathscr{Z}') = [\mathscr{Z}'_{\eta'}] \ ,$$

on the other hand we have

$$(g \times f)^* \mathrm{pr}^*_{\mathscr{G}'}(a) = f^*(a) \ .$$

Simplifying the above expression we get that

$$g^* \circ \mathscr{Z}'_*(a) = \mathscr{Z}_{\eta'*} \circ f^*(a) \ .$$

Thus, the diagram (4.9) is commutative. Since $\mathscr{G}'_{\eta'}$ is nothing but $\Gamma_{\eta'}$, we can also re-write it as

$$\begin{array}{ccc}
A^1(\Gamma_{\eta'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta'}) \\
\Big\uparrow & & \Big\uparrow \\
A^1(\mathscr{G}') & \longrightarrow & A^{p+1}(\mathscr{Y}_{D'})
\end{array} \qquad (4.10)$$

Consider also the obvious commutative diagram

$$\begin{array}{ccc} A^1(\Gamma_{\bar{\eta}}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \\ \uparrow & & \uparrow \\ A^1(\Gamma_{\eta'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta'}) \end{array}$$

Now we come to step IV, in which we understand the group $A^{p+1}(\mathscr{Y}'_D)$.

Step IV:

For that consider the following homomorphism

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\eta'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ ,$$

under the above homomorphism, an element α goes to α' and that is mapped to $\bar{\alpha}$,

$$\alpha \mapsto \alpha' \mapsto \bar{\alpha} \ .$$

The homomorphism $Z_{\bar{\eta}*}$ is onto, so there exists $\bar{\beta}$ such that

$$Z_{\bar{\eta}*}(\bar{\beta}) = \bar{\alpha}$$

Now consider a finite extension L of $k(\gamma_{\eta'})$ such that the cycle $\bar{\beta}$ is defined over L and we have the following commutative triangle,

$$\begin{array}{ccc} A^1(\Gamma_{\eta'}) & \longrightarrow & A^1(\Gamma_{\eta''}) \\ & \searrow & \downarrow \\ & & A^1(\Gamma_{\bar{\eta}}) \end{array}$$

where

$$\Gamma_{\eta''} = \Gamma_{\eta'} \times_{k(\Gamma_{\eta'})} \mathrm{Spec}(L) \ .$$

On the other hand we have the homomorphism

$$A^1(\Gamma_{\eta''}) \to A^1(\Gamma_{\bar{\eta}}) \ .$$

Since $\bar{\beta}$ is defined over $\Gamma_{\eta''}$, there exists β'' that is mapped to $\bar{\beta}$ under the above homomorphism. At the same time consider β' to be the image of β'' under the homomorphism

$$A^1(\Gamma_{\eta''}) \to A^1(\Gamma_{\eta'})$$

which is the push-forward corresponding to the projection

$$\Gamma_{\eta''} \to \Gamma_{\eta'} \ .$$

Now let β'' be mapped to γ'' under the morphism

$$A^1(\Gamma_{\eta''}) \to A^{p+1}(\mathscr{Y}_{\eta''}) \ .$$

Since the homomorphism $A^1(\mathscr{G}') \to A^1(\Gamma_{\eta'})$ is surjective there exists β in $A^1(\mathscr{G}')$ such that it is mapped to β' in $A^1(\Gamma_{\eta'})$. Let β' be mapped to γ', and β mapped to γ under the homomorphisms

$$A^1(\Gamma_{\eta'}) \to A^{P+1}(\mathscr{Y}_{\eta'})$$

and

$$A^1(\mathscr{G}') \to A^{p+1}(\mathscr{Y}_{D'}) \ .$$

Let us show that $\gamma - n_\alpha \alpha$ belongs to the kernel of the homomorphism

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ ,$$

for some positive integer n_α. For that we observe that γ is mapped to γ' and β' is mapped to γ',

$$\beta' \mapsto \gamma', \quad \gamma \mapsto \gamma' \ .$$

Considering the following commutative square

$$\begin{array}{ccc} A^1(\Gamma_{\eta''}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta''}) \\ \downarrow & & \downarrow \\ A^1(\Gamma_{\eta'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta'}) \end{array}$$

we get that γ'' is mapped to γ' under the push-forward from $A^{p+1}(\mathscr{Y}_{\eta''})$ to $A^{p+1}(\mathscr{Y}_{\eta'})$. Therefore γ' is mapped to $N(\gamma'')$ and $n\alpha'$ is mapped to $n\alpha''$ under the pull-back from $A^{p+1}(\mathscr{Y}_{\eta'})$ to $A^{p+1}(\mathscr{Y}_{\eta''})$, where α'' is constructed as follows.

Consider the following commutative triangle

$$\begin{array}{ccc} A^{p+1}(\mathscr{Y}_{\eta'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta''}) \\ & \searrow & \downarrow \\ & & A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \end{array}$$

Let α'' be the element of $A^{p+1}(\mathscr{Y}_{\eta''})$, that is mapped to $\bar{\alpha}$ and the commutativity of the above triangle tells us that α' is mapped to α'' under the pull-back homomorphism

$$A^{p+1}(\mathscr{Y}_{\eta'}) \to A^{p+1}(\mathscr{Y}_{\eta''}) \ .$$

Considering the following commutative square

$$\begin{array}{ccc} A^1(\Gamma_{\bar{\eta}}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \\ \uparrow & & \uparrow \\ A^1(\Gamma_{\eta''}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta''}) \end{array}$$

we obtain that both γ'' and α'' are mapped to $\bar{\alpha}$.

Therefore $\gamma'' - \alpha''$ is a torsion by Lemma 3 in Appendix to Chapter 1 in [7]. Hence there exists an integer m such that

$$m(\gamma'' - \alpha'') = 0 \ .$$

Therefore it follows that

$$m(N(\gamma'') - N(\alpha'')) = 0$$

and $N(\alpha'') = n\alpha''$. Thus we have

$$mn\alpha'' = mN(\gamma'') \ .$$

Therefore $mn\alpha''$ and $mN(\gamma'')$ are mapped to the same element $mn\bar{\alpha}$ under the homomorphism

$$A^{p+1}(\mathscr{Y}_{\eta''}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ .$$

This enforces

$$m\gamma - mn\alpha$$

to be in the kernel of

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ .$$

So taking $m\gamma$ to be γ and mn to be equal to n_α we get that $\gamma - n_\alpha\alpha$ is in the kernel of

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ .$$

Therefore we have

$$n_\alpha\alpha = \gamma + \delta$$

where δ belongs to the kernel of the homomorphism

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \ .$$

Let us now prove that the kernel of the pullback

$$A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \otimes \mathbb{Q}$$

is weakly representable. This is the step V.

Step V :

For that we observe that the homomorphism

$$A^{p+1}(\mathscr{Y}_{\eta'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \otimes \mathbb{Q}$$

is injective, so the kernel of the pullback

$$A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\bar{\eta}}) \otimes \mathbb{Q}$$

coincides with the kernel of

$$A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\eta'}) \otimes \mathbb{Q} .$$

Since $CH^{p+1}(\mathscr{Y}_{\eta'}) \otimes \mathbb{Q}$ is the colimit of the groups $CH^{p+1}(\mathscr{Y}_{W'}) \otimes \mathbb{Q}$, where W' runs over all Zariski open subsets of D', by the localization exact sequence it follows that the kernel

$$A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\eta'}) \otimes \mathbb{Q}$$

is generated by the image of the homomorphisms $r_{t'*}$ from $\oplus_{t' \in D'} A^p(\mathscr{Y}_{t'}) \otimes \mathbb{Q}$ to $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$.

Let us prove it in more detail for the above groups with $\mathbb{Z}$ coefficients. The proof for the groups with $\mathbb{Q}$ coefficients will be the same.

Consider the following commutative diagram.

$$\begin{array}{ccc} A^{p+1}(\mathscr{Y}_{D'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{\eta'}) \\ \downarrow & & \downarrow r \\ CH^{p+1}(\mathscr{Y}_{D'}) & \longrightarrow & CH^{p+1}(\mathscr{Y}_{\eta'}) \end{array}$$

Suppose we take a in the kernel of

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\eta'}) .$$

Then it follows from the above commutative diagram that the image of a under the homomorphism

$$i : A^{p+1}(\mathscr{Y}_{D'}) \to CH^{p+1}(\mathscr{Y}_{D'})$$

is in the kernel of

$$CH^{p+1}(\mathscr{Y}_{D'}) \to CH^{p+1}(\mathscr{Y}_{\eta'}) .$$

Using the fact that $CH^{p+1}(\mathscr{Y}_{\eta'})$ is the colimit of the groups $CH^{p+1}(\mathscr{Y}_{W'})$, where W' is Zariski open in D', we get that the cycle a restricted on $\mathscr{Y}_{W'}$, is zero. That is $i(a)$ is in the kernel of the pullback homomorphism

$$CH^{p+1}(\mathscr{Y}_{D'}) \to CH^{p+1}(\mathscr{Y}_{W'}) .$$

Since pullback preserves algebraic equivalence and we have the commutative diagram

$$\begin{array}{ccc} A^{p+1}(\mathscr{Y}_{D'}) & \longrightarrow & A^{p+1}(\mathscr{Y}_{W'}) \\ \downarrow & & \downarrow r \\ CH^{p+1}(\mathscr{Y}_{D'}) & \longrightarrow & CH^{p+1}(\mathscr{Y}_{W'}) \end{array}$$

we get that a is in the kernel of the homomorphism

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{W'}) \ .$$

The localization exact sequence tells us that, a belongs to the image of

$$\oplus_{t' \in D' \smallsetminus W'} : A^p(\mathscr{Y}_{t'}) \to A^{p+1}(\mathscr{Y}_{D'}) \ .$$

Consequently, the kernel of

$$A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\eta'})$$

is generated by the image of the homomorphism from $\oplus_{t' \in D'} A^p(\mathscr{Y}_{t'})$ to $A^{p+1}(\mathscr{Y}_{D'})$.

As $r_{t*} = 0$ for all but finitely many t in D, it follows that $r_{t'*} = 0$ for all but finitely many t' in D'. The group $A^p(\mathscr{Y}_t) \otimes \mathbb{Q}$ is weakly representable for all $t \in T$ and also $A^p(\widetilde{\mathscr{Y}_t}) \otimes \mathbb{Q}$ is weakly representable for all t such that $\mathscr{Y}_t$ is singular. So we get that the the kernel of the homomorphism

$$A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q} \to A^{p+1}(\mathscr{Y}_{\eta'}) \otimes \mathbb{Q}$$

is weakly representable.

We proved that for any α in $A^{p+1}(\mathscr{Y}_{D'})$, there exists n_α such that $n_\alpha \alpha$ belongs to the subgroup of $A^{p+1}(\mathscr{Y}_{D'})$ generated by the image of $\mathscr{X}'_*$ and the kernel of the pullback $A^{p+1}(\mathscr{Y}_{D'}) \to A^{p+1}(\mathscr{Y}_{\bar{\eta}})$.

Denote this subgroup by B. Tensoring with $\mathbb{Q}$ we get that $B \otimes \mathbb{Q}$ is weakly representable.

From the above it follows that for any α in $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$, there exists n_α such that $n_\alpha \alpha$ is in $B \otimes \mathbb{Q}$. Observe that the group $B \otimes \mathbb{Q}$ is divisible. Therefore we get that there exists b in $B \otimes \mathbb{Q}$, such that

$$n_\alpha b = n_\alpha \alpha$$

or

$$n_\alpha (b - \alpha) = 0 \ .$$

Put

$$\alpha = b - c \ ,$$

where $c = b - \alpha$ and $n_\alpha c = 0$. Since we are working with $\mathbb{Q}$ coefficients it follows that

$$c = 0 \ ,$$

and we have

$$b = \alpha \ .$$

Since b belongs to $B \otimes \mathbb{Q}$, we have that $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$ is a subgroup of $B \otimes \mathbb{Q}$, hence it is equal to $B \otimes \mathbb{Q}$. Therefore, $A^{p+1}(\mathscr{Y}_{D'}) \otimes \mathbb{Q}$ is weakly representable.

Thus, we have got that $A^{p+1}(\mathscr{Y}_{D'})$ is rationally weakly representable. Let us now show that it implies that $A^{p+1}(\mathscr{X})$ is rationally weakly representable, which is the step VI.

Step VI :

Let f be the blow-up morphism from $\mathscr{Y}_{D'}$ to $\mathscr{X}$. By proposition 6.7 b) in [13] we get that f_* is surjective. By the weak representability of $A^{p+1}(\mathscr{Y}_{D'})$ it follows that, there exists a smooth projective curve Σ and an algebraic cycle Z supported on $\Sigma \times \mathscr{Y}_{D'}$ such that the homomorphism Z_* from $A^1(\Sigma)$ to $A^{p+1}(\mathscr{Y}_{D'})$ is onto. Let us show that there exists an algebraic cycle Z' supported on $\Sigma \times \mathscr{X}$ such that Z'_* from $A^1(\Sigma)$ to $A^{p+1}(\mathscr{X})$ is onto. Since f is a proper morphism we can consider $Z' = (\mathrm{id} \times f)_*(Z)$. Now we will prove that the homomorphism Z'_* is onto. For that we want to show that the diagram

$$\begin{array}{ccc} A^1(\Sigma) & \xrightarrow{Z_*} & A^{p+1}(\mathscr{Y}_{D'}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_*} \\ A^1(\Sigma) & \xrightarrow{Z'_*} & A^{p+1}(\mathscr{X}) \end{array}$$

is commutative. So we write

$$f_* Z_*(a) = f_* \mathrm{pr}_{\mathscr{Y}_{D'}*}(Z \cdot \mathrm{pr}_\Sigma^*(a)) \ ,$$

and consider the following diagram

$$\begin{array}{ccc} \Sigma \times \mathscr{Y}_{D'} & \longrightarrow & \mathscr{Y}_{D'} \\ \downarrow & & \downarrow \\ \Sigma \times \mathscr{X} & \longrightarrow & \mathscr{X} \end{array}$$

It gives us that

$$f_* \circ \mathrm{pr}_{\mathscr{Y}_{D'}*} = \mathrm{pr}_{\mathscr{X}*} \circ (\mathrm{id} \times f_*) \ .$$

Putting this in the above formula we get that

$$f_* Z_*(a) = \mathrm{pr}_{\mathscr{X}*} \circ (\mathrm{id} \times f_*)(Z \cdot \mathrm{pr}^*_{\mathscr{Y}_{D'},\Sigma}(a))$$

where $\mathrm{pr}^*_{\mathscr{Y}_{D'},\Sigma}$ is the pullback corresponding to the projection

$$\Sigma \times \mathscr{Y}_{D'} \to \Sigma \ .$$

So we are left with proving

$$\mathrm{pr}_{\mathscr{X}*} \circ (\mathrm{id} \times f_*)(Z \cdot \mathrm{pr}^*_{\mathscr{Y}_{D'},\Sigma}(a))$$

is equal to

$$\mathrm{pr}_{\mathscr{X}*}(Z' \cdot \mathrm{pr}^*_{\mathscr{X},\Sigma}(a)) \ .$$

This is because of the following. Consider Z to be an irreducible subvariety V inside $\Sigma \times \mathscr{Y}_{D'}$. Then $V \cdot \mathrm{pr}^*_{\mathscr{Y}_{D'},\Sigma}(a)$ is nothing but the cycle class of the algebraic cycle associated to

$$V \cap (a \times \mathscr{Y}_{D'})$$

and

$$(\mathrm{id} \times f)(V \cap (a \times \mathscr{Y}_{D'})) = (\mathrm{id} \times f)(V) \cap (a \times \mathscr{X}) \ .$$

Therefore we achieve that

$$(\mathrm{id} \times f)_*(Z \cdot \mathrm{pr}^*_{\mathscr{Y}_{D'},\Sigma}(a)) = Z' \cdot \mathrm{pr}^*_{\mathscr{X},\Sigma}(a) \ ,$$

and hence the diagram

$$\begin{array}{ccc} A^1(\Sigma) & \xrightarrow{Z_*} & A^{p+1}(\mathscr{Y}_{D'}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_*} \\ A^1(\Sigma) & \xrightarrow{Z'_*} & A^{p+1}(\mathscr{X}) \end{array}$$

is commutative. The homomorphism f_* is surjective. The homomorphism $Z_* \otimes \mathbb{Q}$ are is surjective because the group $A^{p+1}(\mathscr{Y}_{D'})$ is rationally weakly representable. This gives that $Z'_* \otimes \mathbb{Q}$ is surjective, and so $A^{p+1}(\mathscr{X})$ is rationally weakly representable. This contradicts to the third assumption of the theorem. Hence, $A_{\bar{\xi},0} = 0$, and Theorem 4.5.2 finishes the proof. □

4.8 Applications to nonsingular cubic fourfolds in $\mathbb{P}^5$

In this last section we consider concrete examples of applications of the above Theorem 4.7.1.

Let us first start with surfaces. A typical example would be the case of $K3$-surfaces over k. So, let $\mathscr{X}$ be a $K3$-surface embedded into $\mathbb{P}^m$. Since hyperplane sections of a surface are curves, Assumptions (A) and (B) are satisfied. Passing to finite extension T' of T we get a section of the morphism $\mathscr{Y}_{T'} \to T'$, which gives that Assumption (A) is satisfied in a family, for $\mathscr{Y}_T$ over T. The third cohomology of a $K3$-surface vanishes and $A^2(\mathscr{X})$ is not representable by Mumford's result in [24]. By Theorem 4.7.1, for a very general hyperplane section $\mathscr{Y}_t$ of the surface $\mathscr{X}$ the kernel of the push-forward homomorphism r_{t*} from $A^1(\mathscr{Y}_t)$ to $A^2(\mathscr{X})$ is countable. It means that, for a fixed point P_t in $\mathscr{Y}_t$, there exists only a countable set of points on the fibre $\mathscr{Y}_t$, which are rationally equivalent to P_t on the surface $\mathscr{X}$. This is a particular case of Proposition 2.4 in [42].

Let us now consider the main case when $\mathscr{X}$ is a nonsingular cubic hypersurface in $\mathbb{P}^5$. It is known that the group $CH^3(\mathscr{X})$ is generated by lines, see [33]. It follows that $A^3(\mathscr{X})$ is generated by differences of linear combinations of line of the same degree. Theorem 4.7.1 tells us something about when two linear combinations of lines of same degree are rationally equivalent to each other.

Let us make sure that cubic fourfolds in $\mathbb{P}^5$ satisfy all the needed assumptions. Since $\mathscr{X}$ is a hypersurface in $\mathbb{P}^5$,

$$H^5_{ét}(\mathscr{X}, \mathbb{Q}_l) = 0 \ ,$$

so that $A_1 = A$. Any smooth hyperplane section $\mathscr{Y}_t$ is a cubic 3-fold in $H_t \simeq \mathbb{P}^4$, whose group $A^2(\mathscr{Y}_t)$ is well known to be representable by the corresponding Prym variety

$$P_t = \mathrm{Prym}(\mathscr{Y}_t)$$

with the corresponding regular isomorphism ψ^2_t between $A^2(\mathscr{Y}_t)$ and P_t, see [5].

Since the Prym construction is of purely algebraic-geometric nature, we can do it over $\bar{\xi}$ getting the Prym variety $P_{\bar{\xi}}$ and the corresponding regular isomorphism $\psi^2_{\bar{\xi}}$ between $A^2(\mathscr{Y}_{\bar{\xi}})$ and $P_{\bar{\xi}}$, for the geometric generic fibre $\mathscr{Y}_{\bar{\xi}}$. In other words, Assumption (A) is satisfied for $\mathscr{Y}_{\bar{\xi}}$. Let us show that Assumption (A) is actually satisfied in a family.

Indeed, we can choose a finitely generated field extension L of the field $k(T)$, such that the fibre $\mathscr{Y}_{\bar{\xi}}$ has a model $\mathscr{Y}_L$ over L, the model $\mathscr{Y}_L$ contains a line Λ, i.e. the closed embedding of Λ into $\mathscr{Y}_{\bar{\xi}}$ is over L, and the Prymian $P_{\bar{\xi}}$ has a model P_L over L. Then choose an appropriate finite extension T' of the scheme T, such that $k(T') = L$, and spread out P_L in to a family of Prymians

$$\mathscr{P} \to W'$$

whose geometric generic fibre is $P_{\bar{\xi}}$ and closed fibres $\mathscr{P}_{t'}$ coincide with P_t if t' is a closed point of W' over a closed point t of U. The line Λ spread into a $\mathbb{P}^1$-bundle over W'. This all gives the consistency of the isomorphisms ψ^2_t and $\psi^2_{\bar{\xi}}$, in the sense that the group-theoretic isomorphisms κ_t between P_t and $P_{\bar{\xi}}$ would coincide with the corresponding scheme-theoretic isomorphism $\varkappa_t$ between P_t and $P_{\bar{\xi}}$.

Assumption (B) is satisfied for $\mathscr{Y}_t$ too. If a hyperplane section $\mathscr{Y}_t$ of the cubic fourfold $\mathscr{X}$ has one ordinary double point, then the singular cubic $\mathscr{Y}_t$ is rational, so that $\tilde{\mathscr{Y}_t}$ is rational. It follows that the group $A^2(\tilde{\mathscr{Y}_t})$ is weakly representable. If $\mathscr{Y}_t$ is nonsingular, then it is unirational and so rationally connected. Hence, $A^3(\mathscr{Y}_t)$ is trivial. The group $A^3(\mathscr{X})$ is not weakly rationally representable by Theorem 0.5 in [31]. Thus, the assumptions of Theorem 4.7.1 are also satisfied.

By Theorem 4.7.1, for each closed point t in the c-open subset U of $\mathbb{P}^{m\vee}$ there exists a countable set Ξ_t of closed points in the Prymian P_t of the hyperplane section $\mathscr{Y}_t$, such that the kernel of the homomorphism r_{t*} from P_t to $A^3(\mathscr{X})$ is countable. In particular, if Σ and Σ' are two linear combinations of lines of the same degree on $\mathscr{X}$, supported on $\mathscr{Y}_t$, then Σ is rationally equivalent to Σ' on $\mathscr{X}$ if and only if the point on P_t, represented by the class of $\Sigma - \Sigma'$, occurs in Ξ_t.

Thus, we have proven the result announced in the Abstract and Introduction above.

Notice also that the group $A^3(\mathscr{Y}_\eta)$ can be non-zero, but we know that it is torsion. Since $A^2(\mathscr{Y}_t)$ is divisible, any cycle class in $A^3(\mathscr{X})$ is represented, up to torsion, by line configurations supported on hyperplane sections.

Bibliography

[1] S. Abhyankar. *On the ramification of algebraic functions.* Am. J. Math. 77, 1955.

[2] S. Abhyankar. *Coverings of algebraic curves.* Am. J. Math. 79, 1957.

[3] J.L. Colliot-Théléne and A.Pirutka. *Hypersurfaces quartiques de dimension 3: non-rationalité stable.* `arxiv:1402.4153v2`, 2014.

[4] K. Banerjee and V. Guletskii. *Rational equivalence for line configurations on cubic hypersurfaces in* $\mathbb{P}^5$. `arXiv:1405.6430v1`, 2014.

[5] A. Beauville. *Variétés de Prym et jacobiennes intermédiaires.* Ann. Sci. Éc. Norm. Sup. 10, 1977, 309 - 391.

[6] S. Bloch. *Torsion algebraic cycles and a theorem of Roitman.* Compositio Math. Volume 39, no. 1, 1979, 107 - 127.

[7] S. Bloch. *Lectures on algebraic cycles*, volume 16 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, second edition, 2010.

[8] S. Bloch and J. Murre. *On the Chow groups of certian types of Fano threefolds.* Compositio Math. 47-105, 1979.

[9] H. Clemens and Ph. Griffiths. *The intermediate Jacobian of the cubic threefold.* Annals of Mathematics 95 (2) pp 281 - 356, 1972.

[10] P. Deligne. *La conjecture de Weil II.* Publications Mathématiques de l'IHÉS 52, 1980, 137 - 252.

[11] B. Fantechi, L. Gottsche, L. Illusie, S. Kleiman, Nitsure, and A. Vistoli. *Fundamental Algebraic Geometry: Grothendieck's FGA Explained.* Mathematical Surveys and Monographs 123, 2005.

[12] E. Freitag and R. Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[13] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, 1984.

[14] S. Gorchinskiy and V. Guletskii. *Motives and representability of algebraic cycles on threefolds over a field.* J. Algebraic Geom. 21, no.2, 2012, 347-373.

[15] A. Grothendieck. *Revetments Etales et Groupe Fundamental(SGA 1)*, volume 224 of *Lecture notes in mathematics,Springer, Berlin, Heidelberg, Newyork*. 1971.

[16] A. Grothendieck and J. Dieudonne. *Étude cohomologique des faisceaux cohérents (EGA III).* Publ.Math.IHES 11(1961), 17(1963).

[17] A. Grothendieck and J. Dieudonne. *Etude locale des schemas et des morphismes des schemas(EGA IV)*, volume Tome 1-4 of *Publ. Math IHES 20(1964), 24(1965), 28(1966), 32(1967).*

[18] V. Guletskii. *Comments on Fulton's intersection theory.* 2013.

[19] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New york-Heidelberg, 1977.

[20] B. Kahn, J. Murre, and C. Pedrini. *On the transcendental part of the motive of a surface*, volume 344 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2007.

[21] S. Kleiman. *The standard conjectures*, volume 55 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994,3-20.

[22] J. Kollár. *Rational curves on algebraic varieties.* Springer, 1996.

[23] J. Milne. *Étale cohomology.* Princeton University Press, 1980.

[24] D. Mumford. *Rational equivalence of* 0*-cycles on surfaces.* J. Math. Kyoto Univ. 9, 1969, 195 - 204.

[25] J. Murre. *Un résultat en théorie des cycles algébriques de codimension deux.* C.R. Acad. Sc. Paris 296, 1983, 981-984.

[26] J. Murre. *Fano varieties and algebraic cycles.* The Fano Conference, pp 51 - 68. Univ. Torino, Turin, 2004.

[27] K. Paranjape. *Cohomological and cycle-theoretic connectivity.* Ann. Math. 140, 1994, 641-660.

[28] A. Roitman. Γ*-equivalence of zero-dimensional cycles (Russian).* Mat. Zbornik. Vol. 86, no. 4, 1971, 557 - 570.

[29] S. Roman. *Field theory*, volume 158 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[30] P. Samuel. *Rational equivalence of arbitrary cycles*, volume 78. Amer. J. Math., 1956, 383-400.

[31] C. Schoen. *On Hodge structures and non-representability of Chow groups*. Compositio Mathematica. Volume 88, Issue 3, 1993, 285-316.

[32] A. Scholl. *Classical motives*. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, Volume 55, 163–187, 1994.

[33] M. Shen. *On relations among* 1*-cycles on cubic hypersurfaces*. J. Alg. Geom. 23, 2014, 539 - 569.

[34] A. Suslin and V. Voevodsky. *Singular homology of abstract algebraic varieties*, volume 123. Invent. Math., 1996,61-94.

[35] A. Suslin and V. Voevodsky. *Relative cycles and Chow sheaves*. In Cycles, Transfers and Motivic Cohomology Theories. Annals of Mathematics Studies 143.,Princeton University Press, 2000.

[36] T. Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2009.

[37] G. Tamme. *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin, 1994.

[38] J.I. Manin V.A. Iskovskih. *Three dimensional quartics and counterexamples to the Luroth problem*, volume 15. Math USSR Sbornik, 1971,141-166.

[39] C. Vial. *Algebraic cycles and fibrations*. Documenta Math. 18, 2013, 1521-1553.

[40] C. Voisin. *Sur les zéro-cycles de certaine hypersurfaces munies d'un automorphisme*. Ann. Scuola Norm. Sup. Pisa Ck. Si. (4) 19, 1992, 473-492.

[41] C. Voisin. *Variations de structure de Hodge et zéro-cycles sur les surfaces générales*. Math. Ann. 299, 1994, 77-103.

[42] C. Voisin. *Symplectic involutions of* $K3$*-surfaces act trivially on* CH_0. Documenta Mathematica 17, 2012, 851-860.

[43] C. Voisin. *Unirational threefolds with no universal codimension 2 cycle*. `arxiv:1312.2122v6`, 2014.